CHAPTER 10

Kinematic characteristics of the highest waves

Estimates of significant or mean wave height, irrespective of the method of their generation, refer to a quasi-stationary interval. The highest wave in a quasi-stationary interval depends on its duration. Other quantiles of wave height distribution can be obtained in deep water using the Rayleigh (I.1) or Forristall (I.2) distributions. In practical applications, it is common to associate the maximum wave height with the estimate of 0.1 % probability. Then, for the Rayleigh distribution $h_{0.1\%}=2.96\bar{h}$. An apparently more efficient approach, however, is comprised of the three following steps. The first step is the estimation of the extreme storm duration for the location of interest. The second step is the calculation of the corresponding probability of the maximum wave (see [Boukhanovsky et al., 1998].) Then, finally, its height can be evaluated.

In shallow water, Glukhovsky's distribution [Glukhovsky, 1966] is used most often in Russia:

$$F(h)=1-exp\left\{-\frac{\pi}{4\left(1+\frac{h^*}{\sqrt{2\pi}}\right)}\left(\frac{h}{\bar{h}}\right)^{\frac{2}{1-h^*}}\right\} \qquad (10.1)$$

where $h^*=\bar{h}/H$ is relative average wave height, and H denotes water depth.

The Rayleigh and Forristall distributions are theoretically unlimited and may predict unrealistically large wave heights. Thus, a limiting height is introduced associated with wave breaking. The extreme height (i.e. the height at which wave breaking occurs) is determined by an equation from the finite amplitude wave theory [Easson, 1997; Sarpkaya, Isaacson, 1981]:

$$h_b/g\tau^2 = C_1 th\left(C_2 H/g\tau^2\right) \qquad (10.2)$$

where h_b is the breaking wave height, g is the acceleration of gravity, H is local water depth, τ is the wave period.

The constants in equation (10.2) are: $C_1=0.02711$ and $C_2=28.77$. Constant C_1 determines the maximum possible steepness of a finite amplitude wave in deep water, while constant C_2 reflects the influence of shallow water effects.

For $H\to 0, h_b=0.78H$.

For $H\to\infty, h_b/\lambda\to 1/7$, λ being the corresponding wavelength.

A more comprehensive description of extreme waves should include not only wave height, but other kinematic characteristics such as period τ, length λ and crest height *c*. The conditional distribution of wave periods for a certain height $F(\tau/h)$, which is also called associated, can be approximated by a Weibull distribution with shape parameters depending on the wave height and period accordingly [Wind, 1974]. The Weibull distribution for the conditional distribution can be written as

$$F(\tau\mid h)=1-exp\left(-A_h\left(\frac{\tau}{m_{\tau|h}}\right)^{k_h}\right),$$
$$A_h=\Gamma^{k_h}\left(1+\frac{1}{k_h}\right) \qquad (10.3)$$

where $m_{\tau|h}$ is the regression between τ and h.

The shape parameter k_h for $\tau|h$ varies from 2.62 to 7.47 [Wind, 1974]. In practice, shape parameters do not depend on the degree of wave development and the type of wave system. The regression

$$m_{x|y}=\int x f(x\mid y)dx$$

is defined as the conditional mean of one random value (the other one is invariant), and the skedastic ratio (conditional variance) is defined as

$$D_{x|y}=\int_{-\infty}^{\infty}\left(x-m_{x|y}\right)^2 f(x\mid y)dx$$

The behaviour of the regression and skedastic ratios for calculated conditional distributions $F(\tau|h)$ is presented in Fig 10.1. Similar figures have been published in many papers (see, e.g., [Wind, 1974; Boukhanovsky et al., 1999; Lopatoukhin, 1974]).

The figure shows that conditional average values of wave periods for a given wave height $m_{\tau|h}$ strongly depend on *h* and τ only for values smaller than the average. For wave height or period exceeding the average, these two parameters become nearly constant. Dependence of conditional variances $D_{\tau|h}$ and $D_{h|\tau}$ on the height and period is seen over the whole range of variability. A parabolic shape of the conditional variance curve indicates that the greatest diversity is pertinent for waves with heights close to the distribution centre. For practical purposes, it may be assumed that mean wave period associated with large waves (at least larger then the mean wave height) is about 1.15-1.20 $\bar{\tau}$. An analysis based on a breakdown of various wave generating conditions would result in a wider probability interval for the scedastic ratio.

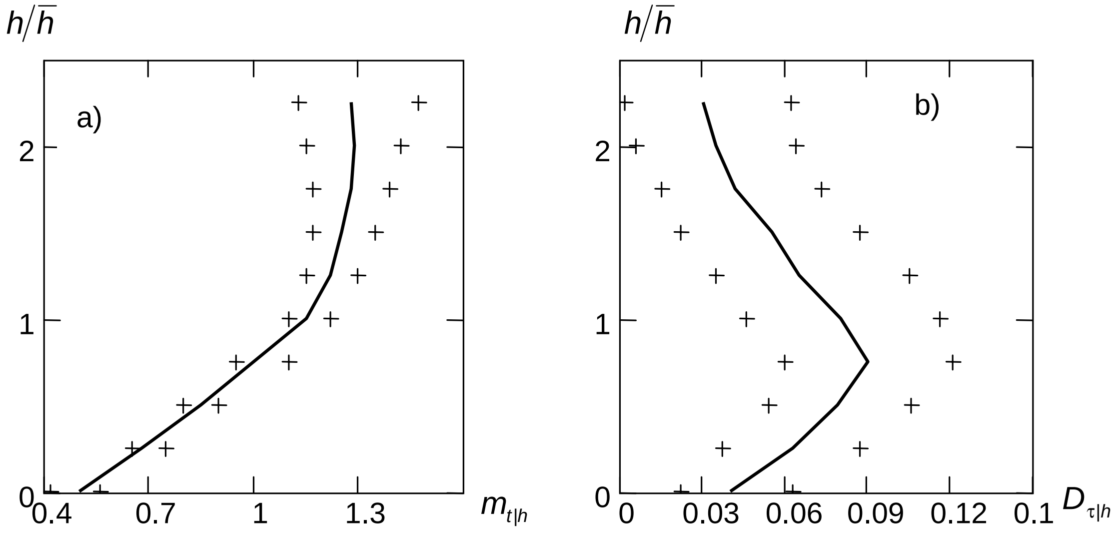

Figure 10.1: Regression (a) and skedastic (b) ratios for the wave periods with prescribed wave height for the Black Sea. Crosses are the borders of 90% confidence interval.

In practical computations of mean period $\bar{\tau}$, which is associated with wave heights h of a certain probability of exceedance, parameterizations of the conditional mean $m_{\tau|h}$ and of its probability range bounds (τl_{ower}, τ_{upper}) are used:

$$\tau(h) = Ah^B \tag{10.4}$$

Davidan et al. (1978) used an extensive array of field observations and suggested the following approximations for the parameters in relation (10.4): *A*=4.8, *B*=0.5.

The lower bound $\bar{\tau}_{lower}$ of mean wave periods depends on wave kinematics and can be determined using an equation for maximal wave steepness. For example, Battjes (1972) proposed the following relation:

$$\bar{\tau}_{lower} = (32\pi h_s / g)^{1/2} \tag{10.5}$$

where h_s stands for significant wave height.

[Teng et al., 1993] used observations from buoys moored off the Atlantic and Pacific coasts of the U.S. and proposed the two following modifications of equation (10.4):

$$\bar{\tau}_{lower} = 3.23 h_s^{0.47} \quad or \quad \bar{\tau}_{lower} = 3.28 h_s^{0.43} \tag{10.6}$$

Other studies (see [Chung-Chu-Teng et al., 1993]) showed that the approximation by Battjes tended to overestimate $\bar{\tau}_{lower}$ while the second formula (10.6) by Teng and co-authors (1993) underestimated it.

The limit on the upper bound $\bar{\tau}_{upper}$ can be defined as a certain quantile of the conditional distribution of wave periods (τ|h) for a given wave height; the 5% probability quantile is a reasonable choice.

Another useful kinematic characteristic of the spectrum is the period τ_p corresponding to its energy peak. Buckley (1988; 1993) and Chun-Chu Teng et al. (1993) used an equation for its lower limit corresponding to a given significant wave height h_s. It is equivalent to the following formula:

$$(\tau_p)_{lower} = 3.62 (h_s)^{0.5} \tag{10.7}$$

For the upper limit $(\tau_p)_{upper}$ Buckley (1988) found the following expression fitting the upper envelope of an empirical data set:

$$(\tau_p)_{upper} = 7.16 (h_s)^{0.5} \tag{10.8}$$

The ratio of period of the spectral peak τ_p to mean period $\bar{\tau}$ is known to vary from 1.1 to 1.4 [Goda, 1979].

Classical hydrodynamics makes it possible to derive all basic kinematic parameters of the wave of interest if parameters (h, τ) are known. For example, the linear theory of small amplitude waves, which is applicable to sufficiently deep waters, yields:

$$\lambda = \frac{g}{2\pi}\tau^2 = 1.56\tau^2, \quad c = h/2 \tag{10.9}$$

For shallow water areas with depth *H*, the theory provides a transcendental equation with parameters *τ, H*

$$\lambda = \tau \sqrt{\frac{g\lambda}{2\pi} tanh\left[\frac{2\pi H}{\lambda}\right]} \qquad (10.10)$$

which can be solved with respect to wavelength λ.

In locations where the water depth is comparable to wave height, it is necessary to use approximations of the potential theory for waves of finite amplitude, e.g. a third-order Stokes expansion for the velocity potential function φ(x,y) (see [Aleshkov, 1996; Sretensky, 1977; Lambrakos et al., 1974)]:

$$\varphi(x,y) = \varepsilon\varphi_1(x,y) + \varepsilon^2\varphi_2(x,y) + \varepsilon^3\varphi_{3(x,y)} + \mathrm{O}(\varepsilon)^4 \qquad (10.11)$$

Here ε is the dimensionless small parameter defined by the kinematic characteristics of the wave, φ_1 is the first expansion term corresponding to a linear approximation of the potential theory of small-amplitude progressive waves, and φ_2 and φ_3 are the non-linear addends corresponding to the second and third approximations.

The free-surface ordinate ζ corresponding to equation (10.11) with accuracy to the third expansion term is written as [Aleshkov, 1996]:

$$\zeta = \left(a + a^3 b_2\right) cos\,\chi + a^2 b_1 \,cos\,2\chi + a^3 b_3 \,cos\,3\chi \qquad (10.12)$$

where *a* is the wave amplitude, χ is the wave phase,

$$b_1 = \frac{k}{4} cthkH \,\frac{2ch^2kH + 1}{sh^2kH}, \quad b_2 = \frac{k^2}{16\,sh^2kH}\left(2ch^6kH + 8ch^4kH - 19ch^2kH + 9\right),$$

$$b_3 = \frac{k^3}{64sh^6kH}\left(1 + 8ch^6kH\right)$$

and *k* is the wave number.

The phase velocity of a third-order Stokes wave is defined as:

$$C = \frac{\omega}{k} = \sqrt{\frac{g}{k} thkH}\left(1 + \frac{a^2k^2}{16\,sh^4kH}\left(8ch^4kH - 8ch^2kH + 9\right)\right) \qquad (10.13)$$

The height of a Stokes wave is equal to:

$$h = 2a\left(1 + a^2k^2b\right) \qquad (10.14)$$

where $b = \frac{1}{64sh^6kH}\left(32ch^6kH + 32ch^4kH - 76ch^2kH + 39\right)$.

The crest height *c* and the trough depth are determined, respectively, for χ=0 and $\chi=\pi$.

The length of a Stokes wave is $\lambda=2\pi/k$, and its amplitude *a* is determined based on the established period $\tau = 2\pi/\omega$ and height *h* by numerical solution of the set of transcendental equations (10.13) and (10.14).

In practice the individual wave crest (c) distribution is provided, for example in [Haring, Heideman, 1980]) as

$$P(c) = 1 - exp\left[-\frac{c^2}{2m_0}\left(1 - B_1\frac{c}{H}\left(B_2 - \frac{c}{H}\right)\right)\right],$$

$$B_1 = 4.37\,,\, B_2 = 0.57 \qquad (10.15)$$

where m_0 is the zeroth moment of wave spectrum. The crest height of p% probability is estimated as a solution to this equation with initial value c_{init}=0.5h_p, in accordance with the analytical solution (10.9) for waves of infinitely small amplitude.

However, to estimate the crest height of waves at n-year return period (*n*=1, 5, 10, 25, 50, and 100 years) for a shallow water area, it is recommended that the crest of a higher-order theory of wave profile is used.

Lambrakos and Brannon (1974) estimated wave crest heights using the higher order Extended Velocity Potential (EXVP) wave theory. The theory considers a Stokes-type wave, which has front-to-back symmetry of its crest and propagates without deformation. The EXVP wave theory is instrumental in determining the geometry and kinematics of individual waves. In the theory the velocity potential has the form

$$\varphi(x,z,t) = \sum cosh(k_n z)[B_n \, sin(k_n x - \omega_n t)] \tag{10.16}$$

In this expression, z is positive upwards from the seafloor, x is positive in the direction of wave propagation, the summation is made over n = 1,2,...,N frequencies, $k_n = 2\pi/\lambda_n$ is wave number for frequency n, $\omega_n = 2\pi/\tau_n$ is angular wave frequency for frequency n, λ_n is wave length for frequency n, and τ_n is wave period for frequency n. The input data for EXVP are the water depth H, the wave height h, and the zero-crossing period τ of the wave.

The tables and plots for the estimation of wave crest are published [Sarpkaya, Isaacson 1981]. A part of those tables is reproduced in Table 10.1.

Let us consider an example. Suppose H=*17.1* m, h=*10.7* m, and τ=12.5 s.

Then from (10.2) we get h_b= 12.8 m, h/h_b=0.83, and $H/g\tau^2$=0.01094. Interpolating data in the Table we obtain: c/h=0.766, i.e. c=8.2 m.

In real situations the ratio c/h varies in the range from 0.50 to 0.80.

Table 10.1

Crest/wave height (c/h) ratio as a function of h/h_b and $H/g\tau^2$

h/h_b	**$H/g\tau^2$**												
	0,0090	**0,0140**	**0,0190**	**0,0240**	**0,0290**	**0,0340**	**0,0390**	**0,0440**	**0,0490**	**0,0540**	**0,0590**	**0,0640**	**0,0690**
0,00	0,5000	0,5000	0,5000	0,5000	0,5000	0,5000	0,5000	0,5000	0,5000	0,5000	0,5000	0,5000	0,5000
0,08	0,5369	0,5262	0,5193	0,5165	0,5145	0,5130	0,5117	0,5109	0,5105	0,5102	0,5098	0,5095	0,5092
0,16	0,5724	0,5509	0,5388	0,5333	0,5294	0,5267	0,5244	0,5229	0,5221	0,5213	0,5206	0,5199	0,5193
0,24	0,6064	0,5751	0,5587	0,5505	0,5447	0,5409	0,5377	0,5356	0,5344	0,5333	0,5322	0,5313	0,5304
0,32	0,6382	0,5994	0,5792	0,5681	0,5604	0,5556	0,5514	0,5488	0,5473	0,5459	0,5447	0,5435	0,5424
0,40	0,6665	0,6234	0,5996	0,5859	0,5764	0,5704	0,5653	0,5622	0,5604	0,5588	0,5574	0,5560	0,5548
0,48	0,6926	0,6468	0,6200	0,6038	0,5925	0,5855	0,5795	0,5758	0,5737	0,5717	0,5700	0,5683	0,5669
0,56	0,7187	0,6698	0,6415	0,6227	0,6095	0,6013	0,5942	0,5898	0,5871	0,5846	0,5824	0,5803	0,5784
0,64	0,7422	0,6934	0,6643	0,6433	0,6283	0,6186	0,6103	0,6049	0,6016	0,5985	0,5957	0,5932	0,5908
0,72	0,7630	0,7178	0,6878	0,6657	0,6493	0,6381	0,6283	0,6221	0,6182	0,6147	0,6114	0,6085	0,6058
0,80	0,7811	0,7407	0,7112	0,6889	0,6718	0,6590	0,6479	0,6410	0,6369	0,6332	0,6298	0,6267	0,6238
0,88	0,7933	0,7564	0,7299	0,7090	0,6924	0,6791	0,6676	0,6604	0,6561	0,6522	0,6486	0,6454	0,6423
0,96	0,7970	0,7614	0,7371	0,7179	0,7031	0,6918	0,6821	0,6756	0,6712	0,6673	0,6636	0,6603	0,6573

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