CHAPTER 4

MENU (MEan Number of Up-crossings) Method

The highest wave h_{max} is observed when a storm is passing the point of observations. To estimate it we have to consider waves as a random process. Let us denote the wave height series as $\xi(t)$. A storm takes place when $\xi(t),\ t\in[t_b,t_e)$ exceeds Z, and h_{max} is the maximum value of $\xi(t) \geq Z$ within interval (t_b,t_e), see Fig.I.1.

The function $\xi(t)$ can cross level Z having positive $\xi'(t) > 0$ or negative $\xi'(t) < 0$ derivative. At the beginning (index b) and ending (index e) times we have $\xi'(t_b) > 0,\ \xi'(t_e) < 0$. At a point of maximum and minimum function $\xi(t)$ has zero derivative $\xi'(t) = 0$, and it reaches its maximum at time t if in the vicinity of this value of t the second derivative is negative, $\xi''(t) < 0$. Thus it is possible to employ the theory of impulses (see [Tikhonov, et all, 1987]) to derive the distribution of extreme values of random process $\xi(t)$. To do so, we need to know the joint distribution density f(ξ,ξ', ξ")[Rice, 1944].

For a stationary Gaussian process $\xi(t)$ with mathematical expectation $m_\xi=0$ and co-variance function $K_\xi(\tau)=\sigma_\xi^2\rho(\tau)$ this distribution reads as follows:

$$f(\xi,\xi',\xi'')=\frac{1}{(2\pi)^{3/2}\sigma_1\sqrt{v}}\exp\left\{-\frac{1}{2v}\left[\sigma_2^2\xi^2+2\sigma_1\xi\xi''+\sigma_\xi^2(\xi'')^2\right]-\frac{2(\xi')^2}{2\sigma_1^2}\right\} \tag{4.1}$$

where σ_1^2 and σ_2^2 are variances of $\xi'(t)$ and $\xi''(t)$, and $v=\sigma_\xi^2\sigma_2^2-\sigma_1^4\geq 0$.

It follows from (4.1) that maximum ξ_m distribution density reads [Longuet-Higgins, 1957]

$$f(\xi_m)=\frac{1}{\sigma_\xi\sqrt{2\pi}}\left[\frac{\sqrt{v}}{\sigma_1\sigma_2}\exp\left(-\frac{\sigma_2^2}{2v}\xi_m^2\right)-\frac{\sigma_1^2\sqrt{2\pi}}{\sigma_\xi^2\sigma_2}\xi_m\exp\left(-\frac{\xi_m^2}{2\sigma_\xi^2}\right)\Phi\left(\frac{\sigma_1^2}{\sigma_\xi\sqrt{v}}\xi_m\right)\right] \tag{4.2}$$

where Φ(x) is the probability integral.

A random process η(t) obeying the log-normal distribution (I.5) can be examined in terms of functionally transformed Gaussian process variables:

$$\eta(t) = \exp[\xi(t)], \qquad \ln\eta(t) = \xi(t) \tag{4.3}$$

Then the maximum distribution density η_m will be:

$$f(\eta_m)=\frac{1}{\sigma\eta_m\sqrt{2\pi}}\left[v\exp\left(-\frac{\ln^2\eta_m}{2\sigma^2v^2}\right)+\sqrt{2\pi(1-v^2)}\frac{\ln\eta_m}{\sigma}\exp\left(-\frac{\ln^2\eta_m}{2\sigma^2}\right)\Phi\left(\frac{\ln\eta_m}{\sigma v}\sqrt{1-v^2}\right)\right] \tag{4.4}$$

Where $v^2=1-\frac{1}{\sigma^2}\left[\frac{K_\eta^{(4)}(0)K_\eta(0)}{(K_\eta''(0))^2}-3\right]^{-1}$,

$$K_\eta(\tau)=\sigma_\eta^2\rho_\eta(\tau)+m_\eta^2=\exp\left[\sigma^2(1+\rho_\xi(\tau))\right];$$

$$m_\eta=\exp\left(\frac{\sigma^2}{2}\right),\quad \sigma_\eta^2=\exp(\sigma^2)\left[\exp(\sigma^2)-1\right],$$

$K_\eta''(0), K^{(4)}(0)$ are second and fourth derivatives of function $K_\eta(\tau)$ at $\tau = 0$.

The largest maximum amongst all ξ_m (or η_m) is called the absolute maximum and we consider it equal to h_{max}. The distribution of h_{max} tends to the same three types of distribution (see (2.2-2.4)) as does the distribution of the maximum value in a sample of independent random values [Leadbetter et al., 1986].

Another possible approach to determination of h_{max} for random process $\xi(t)$ uses dependence of value h on the time t needed to reach this value for the first time (i.e. the first up-crossing). A formal solution to this problem was derived in 1933 by L.S. Pontriagin for the Markov processes only. Nevertheless, there are some relations between the mean value of t and other characteristics of extreme values such as the average number of up-crossings of value h by function $\xi(t)$. Athanassoulis et al., [1995a, 1995b] used these relations to estimate h_{max} on the basis of time series of wave heights taken at standard observation times. The series was represented as periodically correlated stochastic process (PCSP):

$$X(t) = \overline{X}_{tr}(t) + m(t) + s(t)W(t) \qquad (4.5)$$

where the first expansion coefficient represents the linear trend, m is seasonal variation of the parameter *X* mean value, *s* is the seasonal variation of standard deviation, and *W* is a stationary random process (in a general case it is a non-Gaussian process).

This method is called MENU (MEan Number of Up-crossings). It relates extreme value of any wind wave parameter possible once in *T* years to a certain value *x*.* At this value the mathematical expectation *M(x*; t, t+T),* i.e. the mean number of up-crossings of value x* by random process $X(t,\gamma)$ during time interval [*t; t+T],* will be equal to one. Here γ is a sample number such as, say, 1960-1999. For brevity we shall omit parameter γ in further formulae. The process *X(t)* can represent any random parameter related to waves such as height, period, any other distribution parameter, etc. Nevertheless, the most useful application of this method is associated with processing of wave heights or any function related to wave heights (e.g. wave height logarithms).

The function $M(x^*; t_1, t_2)$ can be represented as follows:

$$M(x^*; t_1, t_2) = \int_{t_1}^{t_2}\int_0^{\infty} x' f_{\tau,\tau}(x^*, x')dx'd\tau \qquad (4.6)$$

and therefore its computation requires knowledge of $f_{t_1,t_2}(x,y)$, i.e. the joint distribution density of the process $X(t_1)$ and its derivative $Y(t_2)$, which, in accordance with (4.5), could be expressed through distribution density $f_{W,W'}$.

The approximation of $f_{W,W'}(x,y)$ in [Athanassoulis et al., 1994] was based on the Plackett distribution [Plackett,1965]

$$f(x,y,\psi) = \frac{\psi(\psi-1)(x+y-2xy)+\psi}{\sqrt{(1+(\psi-1)(x+y))^2 - 4\psi(\psi-1)xy}} \qquad (4.7)$$

where parameter ψ is related to the correlation coefficient ρ_{xy} via the following formula:

$$\rho_{xy} = \frac{\psi^2 - 1 - 2\psi\log\psi}{\psi^2 - 1} \qquad (4.8)$$

Correlation coefficient or parameter ψ could be estimated directly from the data or in accordance with maximum likelihood method.
Hence, the MENU method determines extreme values through solving the equation

$$M(x^*; t, t+T) = 1 \qquad (4.9)$$

To solve the integral equation (4.9) in a general case, requires integration with respect to time, for rather long ranges, and taking an improper integral with respect to the other variable *X*. In practice, however, a simplified approach is used, which supposes that *W* is a Gaussian random process [Athanassoulis et al., 1995a].

This approach makes it possible to take the integral with respect to variable *X* analytically. The simplification is valid if the original wave height time series, which is used to compute the extreme wave heights, is distributed log-normally with sufficient accuracy. This is particularly important for the “tail” of the distribution.

Then, assuming that *X(t) = ln(h),* equation (4.9) takes the following form [Athanassoulis et al., 1995a]:

$$\int_t^{t+T(x^*)} Q(\tau)\left(\frac{1}{2C} - \frac{D}{4C}\sqrt{\frac{\pi}{C}}\,exp\left(\frac{D^2}{4C}\right)\right)\left[1 - erf\left(\frac{D}{2\sqrt{C}}\right)\right]exp(-E)d\tau = 1 \qquad (4.10)$$

where

$$C = \frac{\sigma_X^2}{2(\sigma_X^2\sigma_{X'}^2 - \sigma_{XX'}^2)}; \qquad E = \frac{(x^*-m)^2\sigma_{X'}^2 + 2x^*\sigma_{XX'}m' - 2\sigma_{XX'}mm' + \sigma_X^2 m'^2}{2(\sigma_X^2\sigma_{X'}^2 - \sigma_{XX'}^2)};$$

$$D = \frac{\sigma_{XX'}m - \sigma_X^2 m' - x^*\sigma_{XX'}}{\sigma_X^2\sigma_{X'}^2 - \sigma_{XX'}^2}; \qquad Q = \frac{1}{2\pi\sigma_X\sigma_{X'}\sqrt{1-\rho_{XX'}^2}}; \rho = \frac{\sigma_{XX'}}{\sigma_X\sigma_{X'}}.$$

In the above relations *x** denotes the up-crossing level (the wave height) occurring once in time *T*. Functions σ and *m* are, respectively, the standard deviation and mathematical expectation that can be obtained from (4.5) using an approximation with low order Fourier expansion. The standard error function is called “erf”, as usual. Therefore, if we have any multi-year time series $\{X(t), 0 \le t \le Tend\}$ where *X* may be *h(t)* or *ln(h(t)+c),* then the MENU method can lead to the following simplified procedure of estimation of return period *T* associated with the level *x*.* The procedure involves several steps [Athanassoulis,1995, Stephanakos, 1999]:

1. Reindexing of X(t) series as follows:

$$\left\{X\left(j,t^{\alpha}\right), j=1,2,\ldots,J, \quad 0\le t^{\alpha}\le T^{\alpha}\right\}$$

T^{α} = 365 × 24 (hours) is one year

$t_k^{\alpha}=\left(k-\tfrac{1}{2}\right)\Delta t, \quad k=1,2,\ldots,K=T^{\alpha}/\Delta t$ where Δt is the sampling interval.

2. Making the time series trend-free

   a) Obtaining the sequence of mean annual values

$$\overline{X}(j)=\frac{1}{K}\sum_{k=1}^{K}X\left(j,t_k^{\alpha}\right)$$

   b) Fitting the data points to linear function

$$X_{tr}(t)=B_0+B_1 t/T^{\alpha}$$

   c) Deleting the trend $Y(t)=X(t)-\overline{X}_{tr}(t)$

3. Derivation of seasonal characteristic

$$m\left(t_k^{\alpha}\right)=\frac{1}{J}\sum_{j=1}^{J}Y\left(j,t_k^{\alpha}\right),$$
$$s\left(t_k^{\alpha}\right)=\sqrt{\frac{1}{J}\sum_{j=1}^{J}\left[Y\left(j,t_k^{\alpha}\right)-m\left(t_k^{\alpha}\right)\right]^2} \qquad (4.11)$$

4. Low-order Fourier series representation for *m* (order μ – *1*) and *s* (order σ – *3*). See Fig.4.1.

5. Time series decomposition in accordance with (4.5)

$$W\left(j,t^{\alpha}\right)=\frac{Y\left(j,t^{\alpha}\right)-\mu\left(t^{\alpha}\right)}{\sigma\left(t^{\alpha}\right)} \qquad (4.12)$$

6. Obtaining the joint probability density f(X,X')

$$X'(t)=B_1/T^{\alpha}+\mu'(t)+\sigma'(t)W(t)+\sigma(t)W'(t) \qquad (4.13)$$

Considering the density of (W,W'), there are several ways to model it:

   a) Use bi-variate Plackett model (4.7) - (4.8) with uni-variate marginal densities estimated from the uni-variate samples of *W(t)* and *W'(t)*, respectively.

   b) Use the bi-variate Plackett model with both uni-variate marginals log-normal for the joint density of *(W(t)*, *W(t+Δt))*, and by means of bi-variate linear transformation

   W(t) = W(t); W'(t)=(W(t+Δt)-W(t))/Δt,

   calculate the $f^{WW'}$

   c) Assume that the joint density $f_{t,t+\Delta t}^{WW}$ is a bi-variate normal density, and, by means of previous linear transformation, obtain the density $f_{t,t}^{WW'}$

7. Calculating the coefficients in (4.10) with the help of (4.5), (4.13) and low-order Fourier expansions for *m* and *s*.

8. Numerical solution of equation (4.10) for given level *X** that yields the return period *T(X*)*.

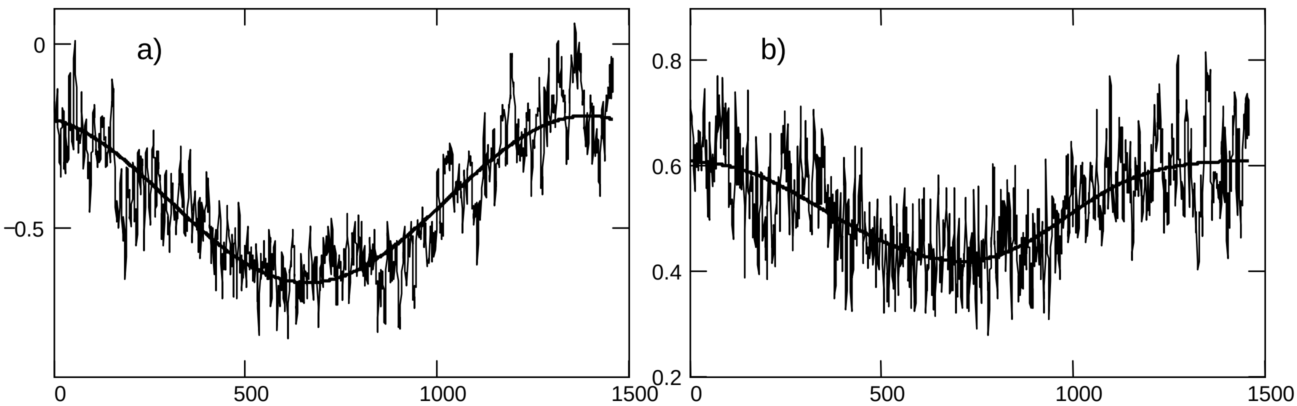

Figure.4.1. Seasonal variability of ln h1/3 time series for the Baltic Sea.
(a) Seasonal mean value *m*(t^{α}) and its 1st order Fourier representation (b) Seasonal standard deviation *s*(t^{α}) and its 3rd order Fourier representation. Axis of abscissa is annual time t^{α} (6-hour intervals)

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