CHAPTER 3

Joint distribution of wave directions and heights

If a wave field is comprised of several wave patterns, then each of the patterns /wind sea or swell(s)/ can be described by a corresponding wave height *h*, period τ, for example $\bar{h}, \bar{\tau}$ and direction of propagation β.

The joint distribution density *f(h, β)* is used most frequently for the probabilistic description of the wave field. An example of corresponding recurrence *f(h,β)dhdβ* (%) for the Baltic Sea (autumn) is given in Table 3.1 and is shown in Fig. 3.1.

The analysis of such joint distributions is specific because *(h,β)* represents a system of random values where h is a scalar value and *β* is an angular value. The coefficient of colligation C_f will be used to check the hypothesis that random values *h* and *β* are independent, as follows:

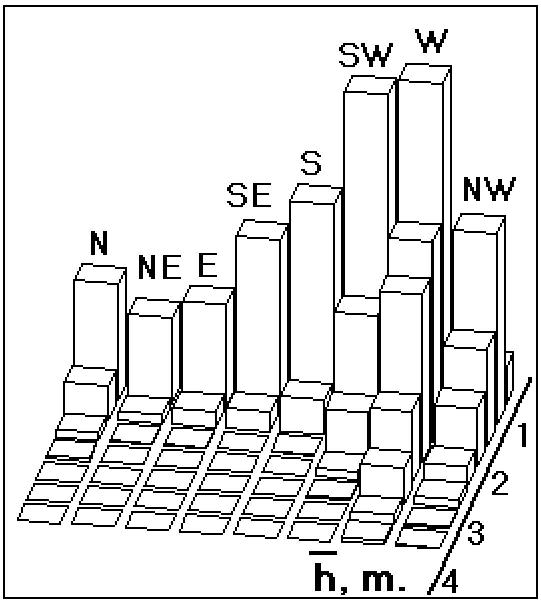

Figure 3.1. Probability of wave heights by direction The Baltic Sea

Table 3.1.
Joint probability (%) of mean wave heights and directions.
The Baltic Sea. Autumn. Total number of observations N=12770.

h, m	N	NE	E	SE	S	SW	W	NW	f(h) dh
0.0-0.5	1	1	2	3	4	4	3	1	19
0.5-1.0	4	3	3	5	7	10	11	6	49
1.0-1.5	1	<1	1	1	1	4	6	3	18
1.5-2.0	<1	<1	<1	–	<1	1	5	2	9
2.0-2.5	<1	–	–	–	–	<1	2	1	3
2.5-3.0	–	–	–	–	–	<1	1	<1	1
>3.0	–	–	–	–	–	–	<1	<1	<1
f(β)dβ	6	4	6	9	12	20	29	14	100

$$C_f = \frac{f(h,\beta)}{f(h)f(\beta)} \qquad (3.1)$$

Also we will use regression lines:

$$m_{h|b} = \int_0^{\infty} h\, f(h \mid \beta)dh \qquad (3.2)$$

$$m_{\beta|h} = \int_0^{2\pi} \beta\, f(\beta \mid h)d\beta \qquad (3.3)$$

and scedastic (conditional variance) curves:

$$D_{h|\beta} = \int_0^{\infty} (h - m_{h|\beta})^2 f(h \mid \beta)dh \quad , \qquad (3.4)$$

$$D_{\beta|h} = \int_0^{2\pi} (\beta - m_{\beta|h})^2 f(\beta \mid h)d\beta \qquad (3.5)$$

According to [Mardia, 1972] the regression is estimated as the mean direction

$$\bar{\beta} = arctan\left[\bar{S}/\bar{C}\right] \qquad (3.6)$$

while expression

$$D_\beta = 1 - R \qquad (3.7)$$

with

$$\bar{C} = \frac{1}{n}\sum_{i=1}^{n} cos\,\beta_i\,,\ \bar{S} = \frac{1}{n}\sum_{i=1}^{n} sin\,\beta_i\,,\quad R = \sqrt{\bar{C}^2 + \bar{S}^2}$$

gives an estimate for conditional variance (3.5). The value

$$\sigma = \sqrt{-2\ln(1-D_\beta)} \quad (3.8)$$

that follows from an analogy with a wrapped normal distribution can be used as a measure of spreading of angular value β. It is similar to the r.m.s. deviation. Tables 3.2 and 3.3 are based on the data from Table 3.1, and present conditional means /see (3.2),(3.3)/ and variances /see (3.4),(3.5)/ for height of waves from different directions.

Table 3.2.

Conditional means (3.2), variance (3.4), and r.m.s. deviation of wave height h for given direction β

β	N	NE	E	SE	S	SW	W	NW
$m_{h\|\beta}$, m	0.8	0.7	0.6	0.6	0.6	0.8	1.3	1.1
$D_{h\|\beta}$, m^2	0.1	0.1	0.1	0.1	0.1	0.2	0.5	0.4
$\sigma_{h\|\beta}$, m	0.4	0.3	0.3	0.3	0.3	0.5	0.7	0.6

Table 3.3.

Conditional means (3.3), variance (3.5), and r.m.s. deviation of wave direction β for given wave height h

h, m	0.25	0.75	1.25	1.75	2.25	2.75	3.25	3.75
$m_{\beta\|h}$, ^{0}	189	239	264	273	276	275	275	275
$D_{\beta\|h}$, rad^2.	0.60	0.68	0.38	0.13	0.09	0.05	0.03	0.03
$\sigma_{\beta\|h}$, ^{0}	78	86	56	30	25	18	14	13

The computations show that values of c_f differ from 1, lines of regression (3.2) and (3.3) are not parallel to co-ordinates, conditional variance $D_{\beta|h}$ differs from the absolute variance D_β = 0.59 rad^2,.σ_β = 76°. This suggests that the two-dimensional distribution density f(h,β) has to be approximated by expression

$$f(h,\beta) = f(h) f(\beta|h) \quad (3.9)$$

Among existing approximations of f(β│h) the most useful is the Mises distribution. Its mean μ and scale *k* parameters depend on wave height *h*:

$$f(\beta|h) = \frac{1}{2\pi I_o(k(h))} exp[k(h)cos[\beta - \mu(h)]], \quad (3.10)$$

where $|\mu(h)| < \infty, \quad k(h) > 0$.

Here

$$I_0(k) = \sum_{r=0}^{\infty} \frac{1}{(r!)^2}\left[\frac{k}{2}\right]^{2r} \quad (3.11)$$

is the modified first type zero-order cylindrical function. Variation of parameters μ and *k* makes it possible to reconstruct all angular distributions, from uniform to a narrow one.

Comparisons of f(h,β) and approximation (3.10) for parameters μ and *k* taken from Table 3.4 showed satisfactory agreement.

We can use the distribution F(h|β) for any given β to estimate h_{max} with the help of both methods of initial distribution and the annual maximum series. The absolute wave height distribution is a mixed distribution of waves coming from different sectors of wave directions β_i:

$$f(h) = \sum_i \gamma_i f(h \mid \beta_i). \quad (3.12)$$

Here γ_i are weight coefficients, which meet the compliance condition that $\sum_i \gamma_i = 1$. Computation of extreme values for individual directions is then made by simple adjustment of omni-directional wave height distribution *f(h)* with distribution of wave height for given directions $f(h|\beta_i)$. Wave height *h(T)*, expected to occur once in *T* years for direction β will be equal to a certain quantile of the omni-directional distribution *f(h).*

Table 3.5 provides estimates of a hundred year mean wave height, both directional and omni-directional, based on data from Table 3.1.

Table 3.4.

Parameters μ and k of conditional distribution of wave heights (3.10). Data from Table 3.1

h, m	0–0.5	0.5–1.0	1.0–1.5	1.5–2.0	2.0–2.5	2.5–3.0	3.0–4.0
μ, ^{0}	189	239	264	273	276	275	275
k	0.8	0.7	1.8	4.2	6.5	7.2	7.8

Table 3.5

Directional and omni-directional estimates of one hundred year wave height. AMS method. The Baltic Sea

β	**N**	**NE**	**E**	**SE**	**S**	**SW**	**W**	**NW**	**omni-directional**
h(100), m	2.7	2.4	2.3	2.4	2.7	4.0	5.1	4.4	5.1

It is seen that the highest waves in the Baltic Sea mostly propagate from the west. If traditional computations of absolute and conditional distributions are used, a discrepancy often occurs that the omni-directional wave height estimate exceeds the maximum (over all directions) estimate computed taking into account directional distribution [Proceedings, 1986]. Estimates of long return period wave heights, if they are based on relation (3.12), eliminate this typical discrepancy.

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