QUASI-RESONANT INTERACTIONS IN SHALLOW WATER

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OBJECTIVES

- **STUDY NON-LINEAR INTERACTIONS IN SHALLOW WATER FROM A STATISTICAL POINT OF VIEW**

- **DEVELOP A WAVE ACTION BALANCE EQUATION TO BE APPLIED IN SHALLOW WATER**

- **PREDICT THE STATISTICAL PROPERTIES OF SURFACE ELEVATION (not just wave spectra but also skewness) IN SHALLOW WATER**

MOTIVATIONS

•"…***Generally speaking, the weakly nonlinear theory has narrow frames of applicability in shallow water***…" (from Zakharov 1999, Eur. J. Mech. B/Fluids)

• **Recent paper by Janssen (JPO 2003) in which the role of quasi-resonant interactions in deep water are addressed**

LIMITATIONS OF THE KINETIC EQUATION

The standard kinetic equation describes the evolution of free waves

- **While in deep water the contribution of bound waves can be neglected, as the water depth decreases bound waves become important**

- **In order to derive the kinetic equation, three-wave interactions are removed by a canonical transformation. In the shallow water limit, this transformation reduces to the Stokes expansion which converges only if**

$$Ur = ak/(kh)^3 << 1$$

DETERMINISTIC THREE-WAVE INTERACTION EQUATION

$$\frac{\partial a_0}{\partial t} + i\omega_0 a_0 + i\int U^{(1)}_{0,1,2} a_1 a_2 \delta(\mathrm{k}_0 - \mathrm{k}_1 - \mathrm{k}_2) d\,\mathrm{k}_{1,2} +$$

$$+2i\int U^{(1)}_{2,1,0} a_1^* a_2 \delta(\mathrm{k}_0 + \mathrm{k}_1 - \mathrm{k}_2) d\,\mathrm{k}_{1,2} + i\int U^{(3)}_{0,1,2} a_1^* a_2^* \delta(\mathrm{k}_1 + \mathrm{k}_2 + \mathrm{k}_0) d\,\mathrm{k}_{1,2} = 0$$

with $a_0 = a(k_0, t)$

The surface elevation and the velocity potential are related to the complex variable $a(k,t)$ **as follows:**

$$\eta(k) = \sqrt{\frac{\omega(k)}{2g}}(a(k) + a^*(-k)) \qquad \psi(k) = \sqrt{\frac{g}{2\omega(k)}}(a(k) - a^*(-k))$$

The goal is to write an evolution equation for the wave action spectrum:

- **Homogeneity:**

$$< a(k_0)a^*(k_1) >= N(k_0)\delta(k_0 - k_1)$$

- **Quasi Gaussian Approximation:**

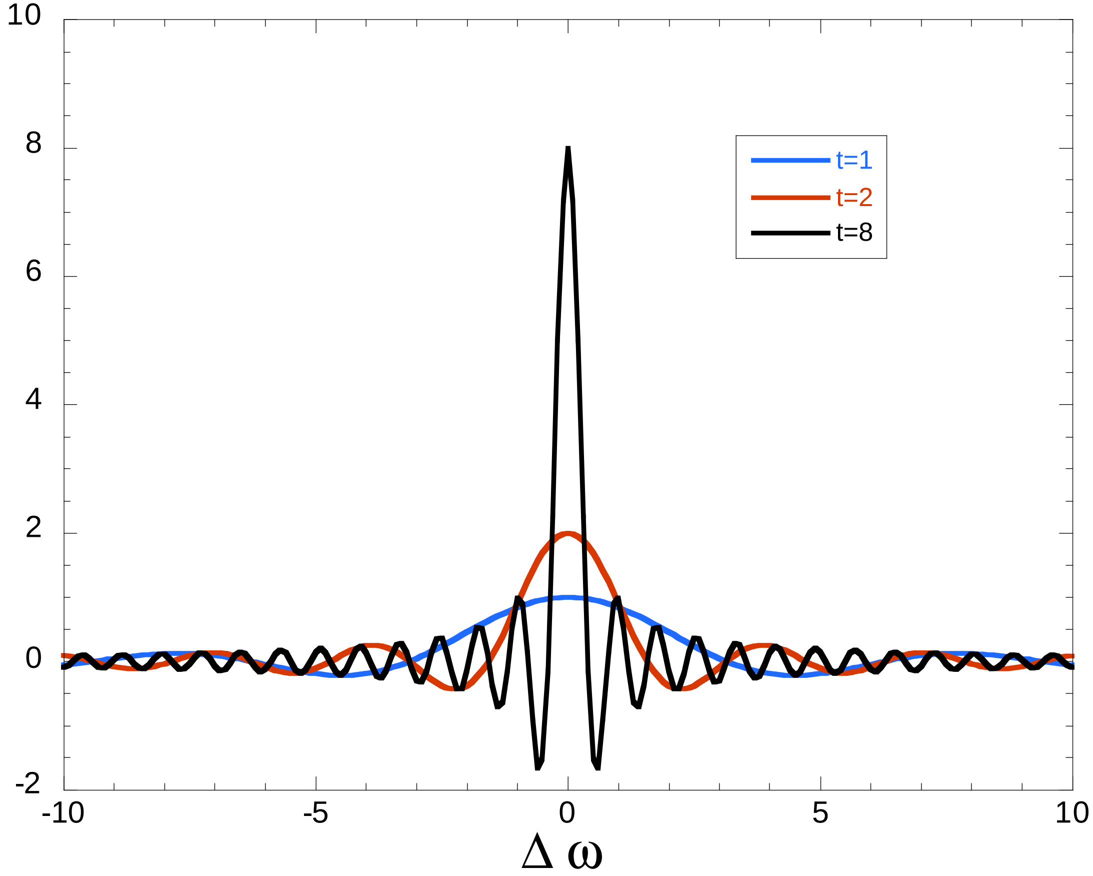
$$< a_1 a_2 a_3^* a_4^* >= N_1 N_2 (\delta(k_1 - k_3)\delta(k_2 - k_4) + \delta(k_1 - k_4)\delta(k_2 - k_3))$$

- **We end up with two coupled equations: one for the wave spectrum and the second for the bi-spectrum. The equation for the bi-spectrum is integrated analytically and the system is reduced to a single evolution equation for the wave action spetrum:**

$$\frac{\partial N_0}{\partial t} = 4\int \left|U_{0,1,2}\right|^2 [(N_1 N_2 - N_1 N_0 - N_2 N_0]\delta(\mathrm{k}_1 + \mathrm{k}_2 - \mathrm{k}_0)\frac{Sin[(\omega_1 + \omega_2 - \omega_0)t]}{\omega_1 + \omega_2 - \omega_0} d\,\mathrm{k}_{1,2} -$$

$$8\int \left|U_{2,1,0}\right|^2 [(N_1 N_0 - N_1 N_2 - N_0 N_2]\delta(\mathrm{k}_2 - \mathrm{k}_1 - \mathrm{k}_0)\frac{Sin[(\omega_2 - \omega_1 - \omega_0)t]}{\omega_1 - \omega_2 - \omega_0} d\,\mathrm{k}_{1,2}$$

THE SPREAD DELTA FUNCTION



For large times:

$$\lim_{t\to\infty} \frac{Sin[(\omega_1 + \omega_2 - \omega_3)\; t]}{\omega_1 + \omega_2 - \omega_3} = \pi\delta(\omega_1 + \omega_2 - \omega_3)$$

In the shallow water limit the equation becomes:

$$\frac{\partial P(k,t)}{\partial t} = \frac{9}{4}\left(\frac{c_0}{h}\right)^2 k \int [(kP_1P_2 - k_2P_1P - k_1P_2P]\delta(\mathrm{k}_1 + \mathrm{k}_2 - \mathrm{k}_0)\frac{Sin[\Delta\omega t]}{\Delta\omega} d\,\mathrm{k}_1 d\,\mathrm{k}_2$$

Skewness can also be estimated directly form the wave spectrum:

$$skewness = \frac{<\eta^3>}{<\eta^2>^{3/2}} =$$

$$= \frac{1}{m_0^{3/2}}\frac{3}{2}\frac{c_0}{h}\int [(kP_1P_2 - k_2P_1P - k_1P_2P]\delta(\mathrm{k}_1 + \mathrm{k}_2 - \mathrm{k}_0)(1 - Cos[\Delta\omega t])d\,\mathrm{k}_1 d\,\mathrm{k}_2\, d\,\mathrm{k}_3 \approx$$

$$\approx Ur(1 - Cos[c_0 k_p^3 h^2 t])$$

PRELIMINARY TEST OF THE STOCHASTIC EQUATION FOR WAVE SPECTRA:

Compare spectra and skewness from numerical simulations from the stochastic equation with numerical simulations from ensemble deterministic equation

We will consider a standard Jonswap spectrum and use it as a boundary condition for our simulations and look at the evolution in space of spectra and skewness.

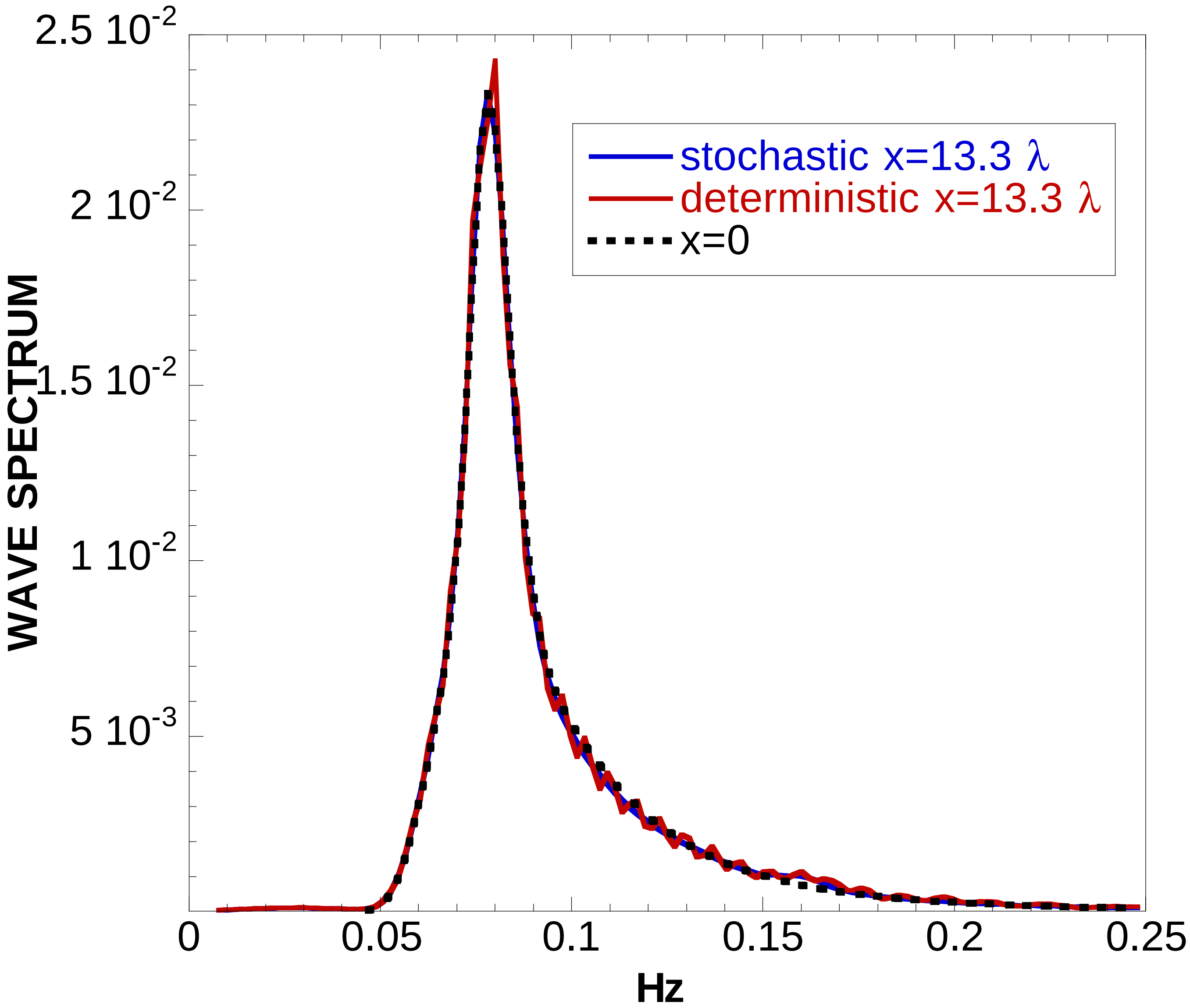
Three different conditions are considered:

$Ur = 0.11$ $kh=0.34$

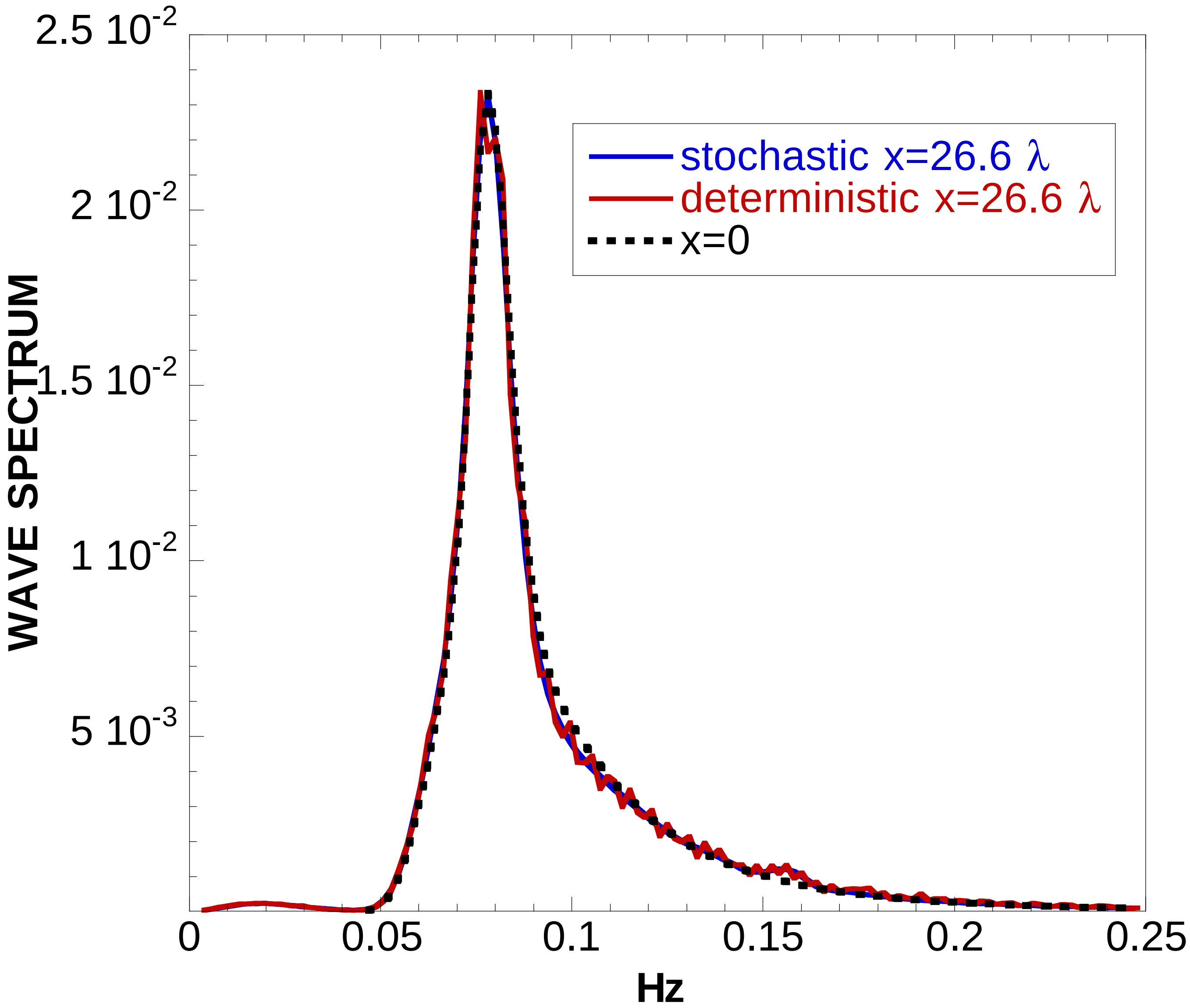
$Ur = 0.20$ $kh=0.29$

$Ur = 0.46$ $kh=0.23$

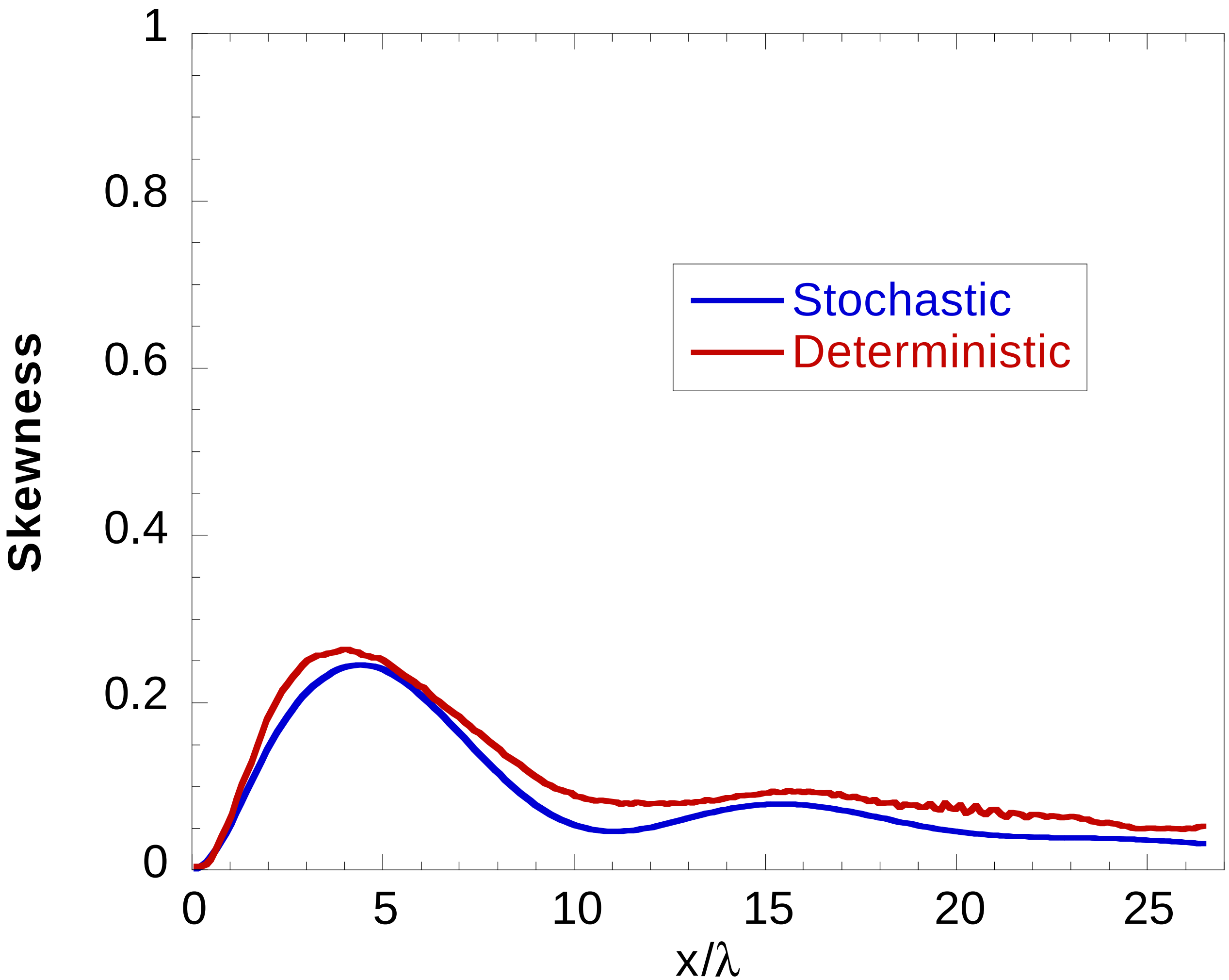
$Ur = 0.11$



2.5 10-2
2 10-2
1.5 10-2
1 10-2
5 10-3
WAVE SPECTRUM
0
0.05
0.1
0.15
0.2
0.25
Hz
stochastic x=13.3 λ
deterministic x=13.3 λ
x=0

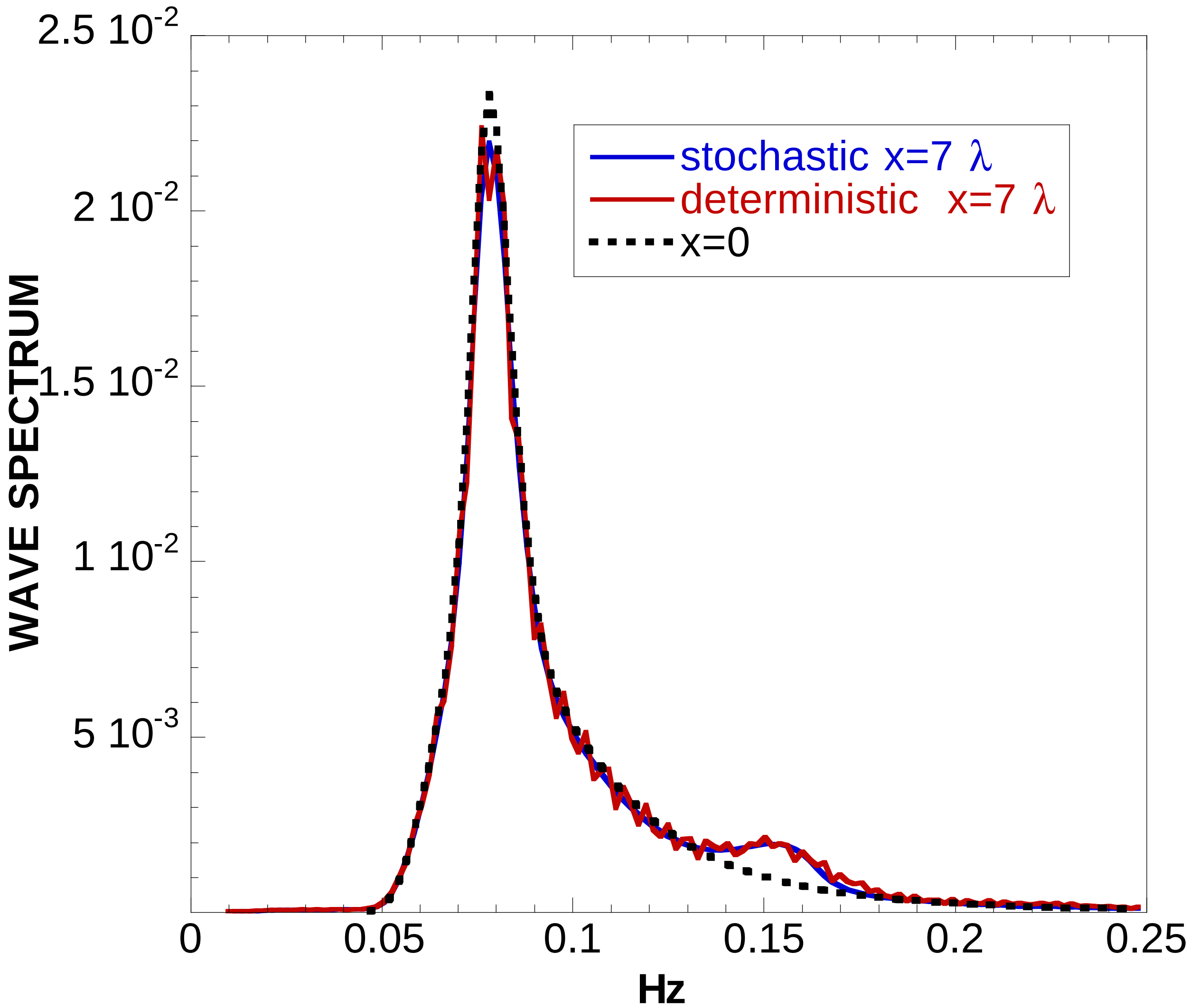


stochastic x=26.6 λ
deterministic x=26.6 λ
x=0
WAVE SPECTRUM
Hz
2.5 10-2
2 10-2
1.5 10-2
1 10-2
5 10-3
0
0.05
0.1
0.15
0.2
0.25

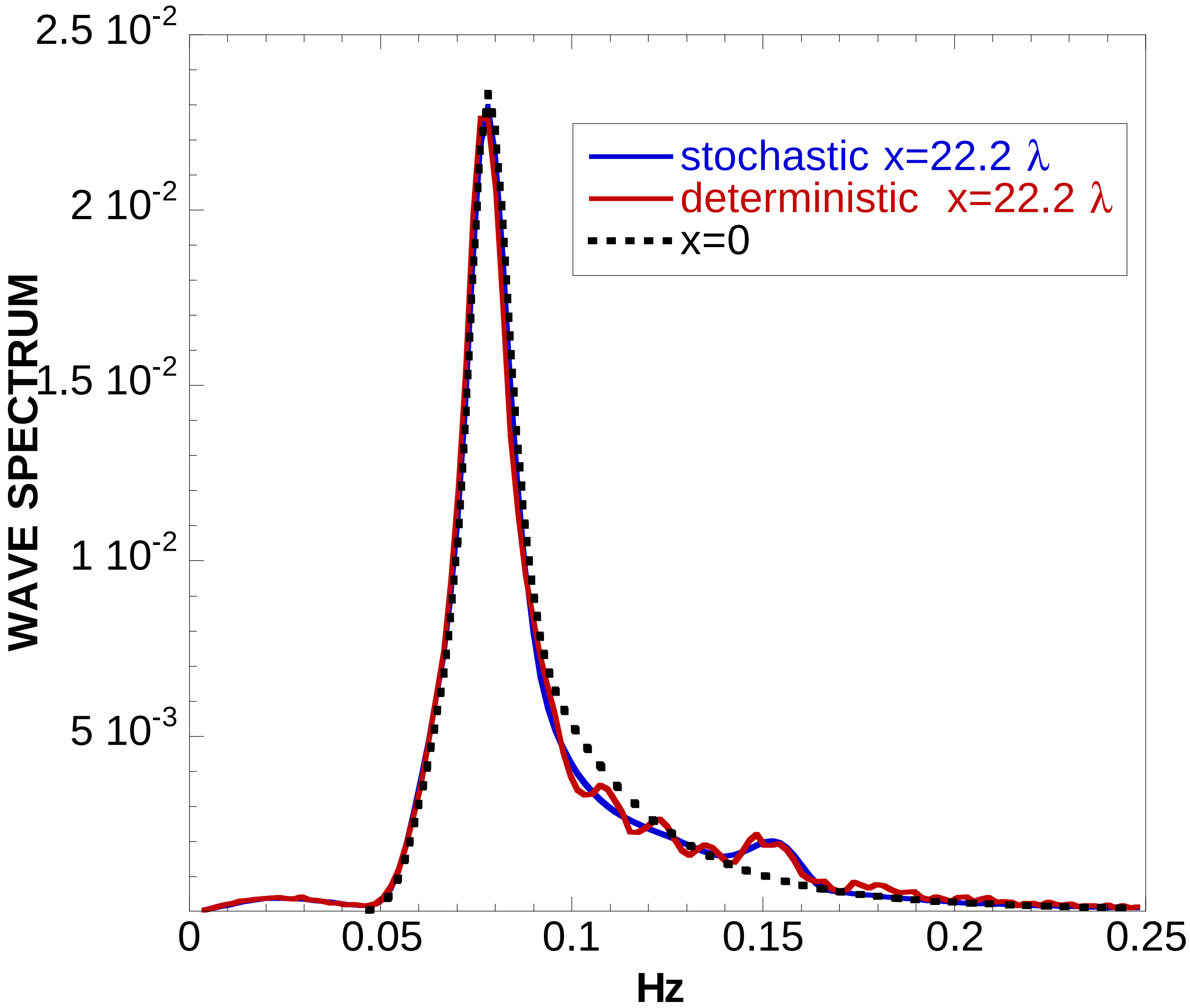


Stochastic
Deterministic
Skewness
x/λ
1
0.8
0.6
0.4
0.2
0
0
5
10
15
20
25

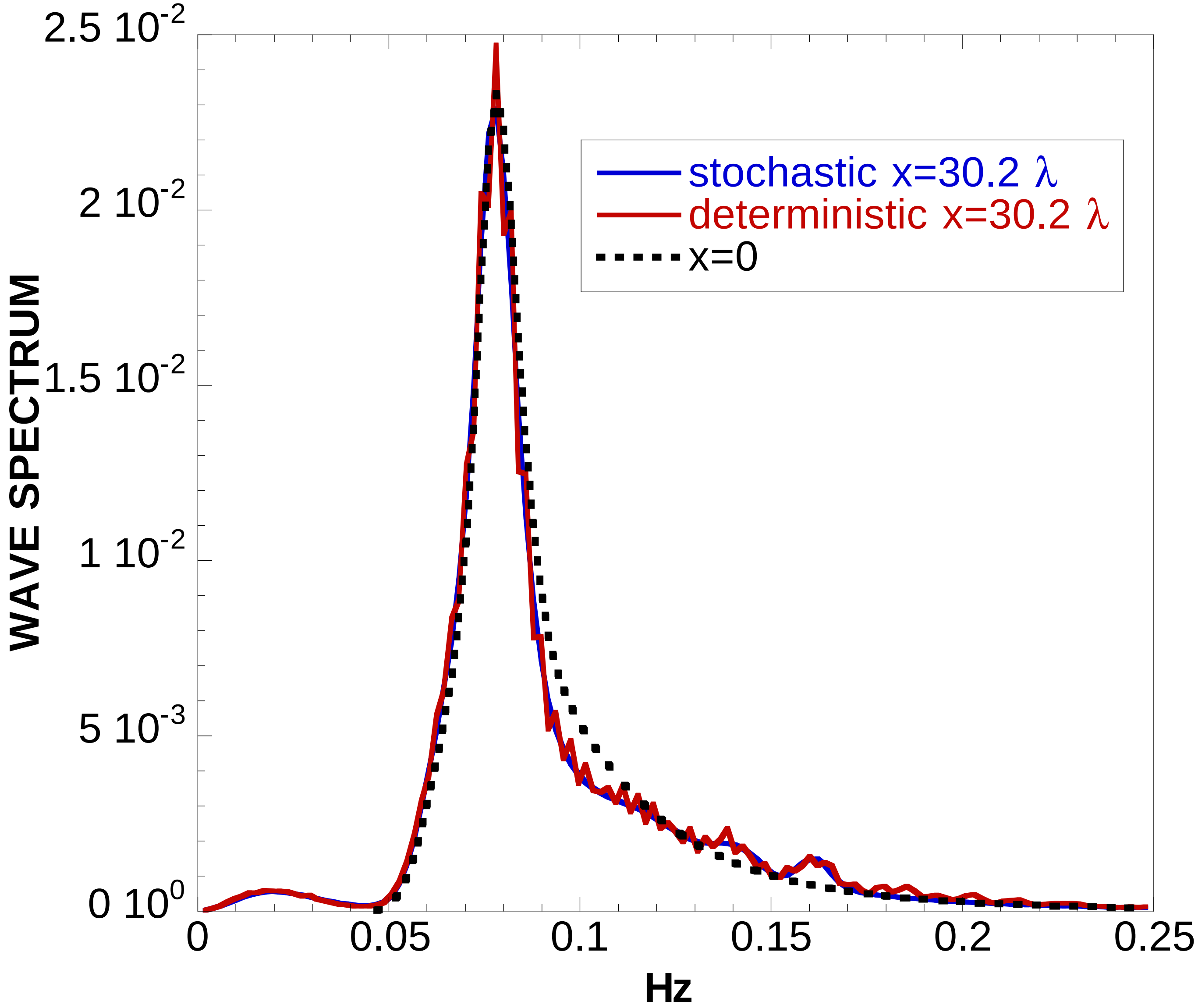
Ur=0.2



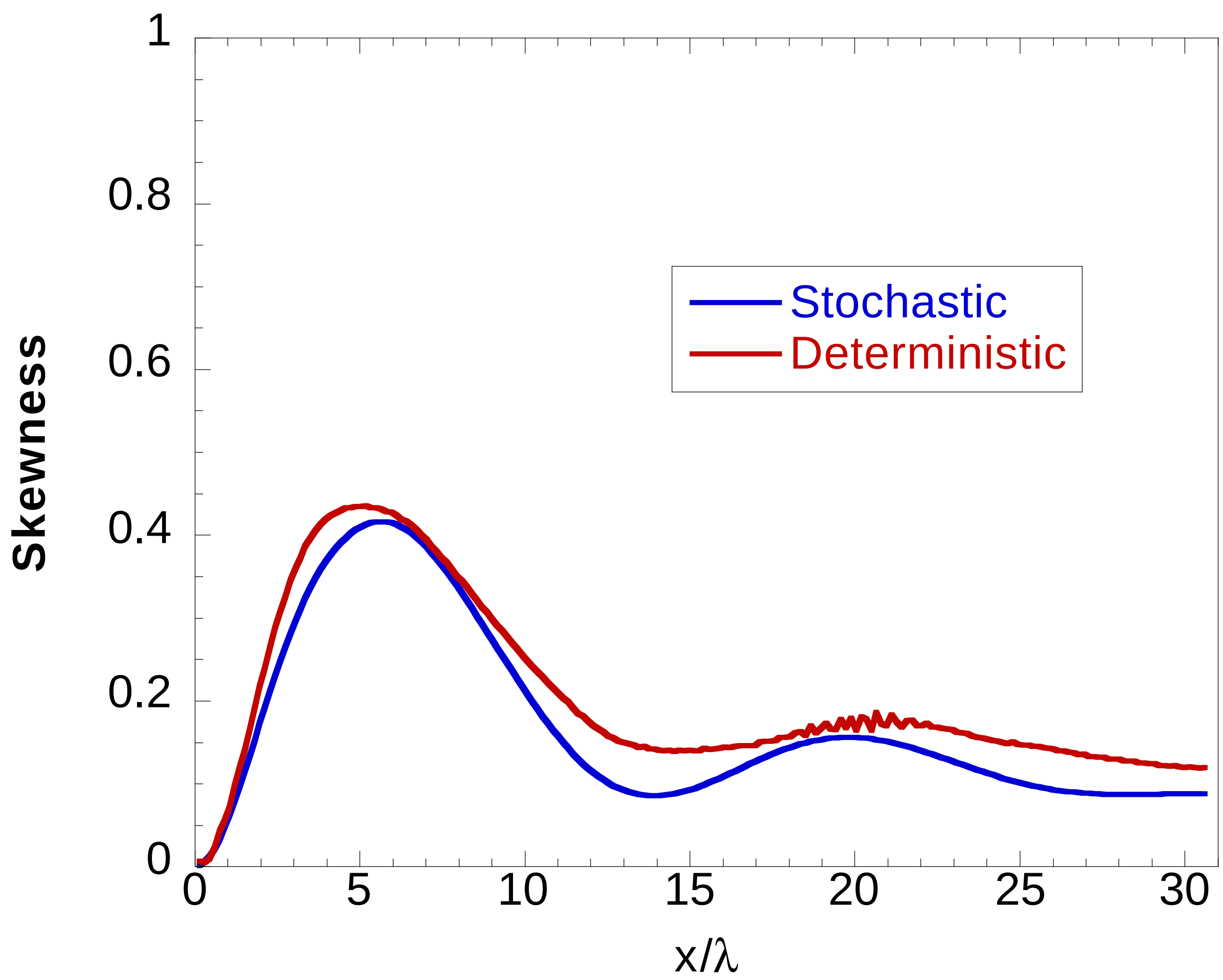
2.5 10-2
2 10-2
1.5 10-2
1 10-2
5 10-3
WAVE SPECTRUM
0
0.05
0.1
0.15
0.2
0.25
Hz
stochastic x=7 λ
deterministic x=7 λ
x=0



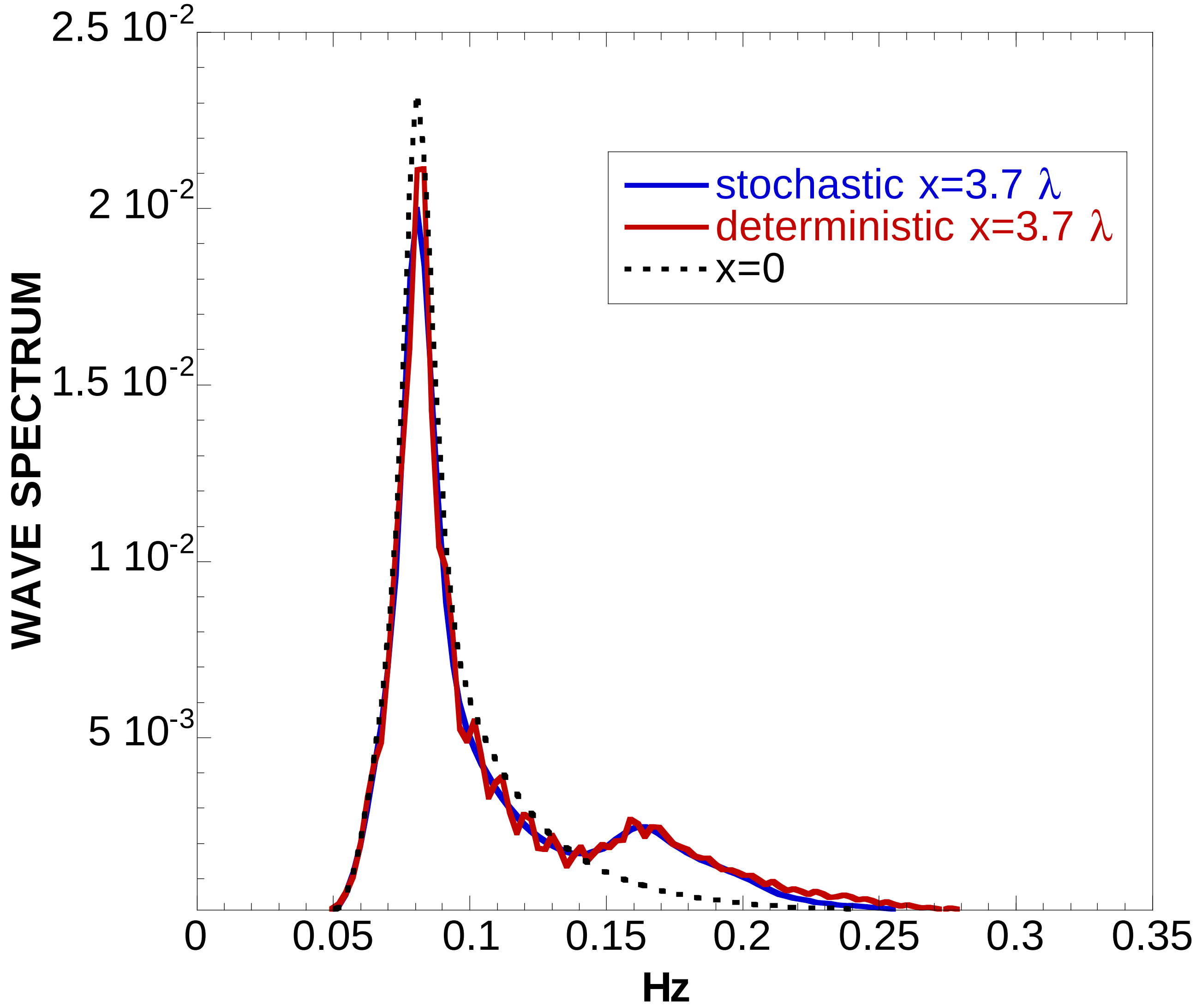
2.5 10-2
2 10-2
1.5 10-2
1 10-2
5 10-3
WAVE SPECTRUM
0
0.05
0.1
0.15
0.2
0.25
Hz
stochastic x=22.2 λ
deterministic x=22.2 λ
x=0



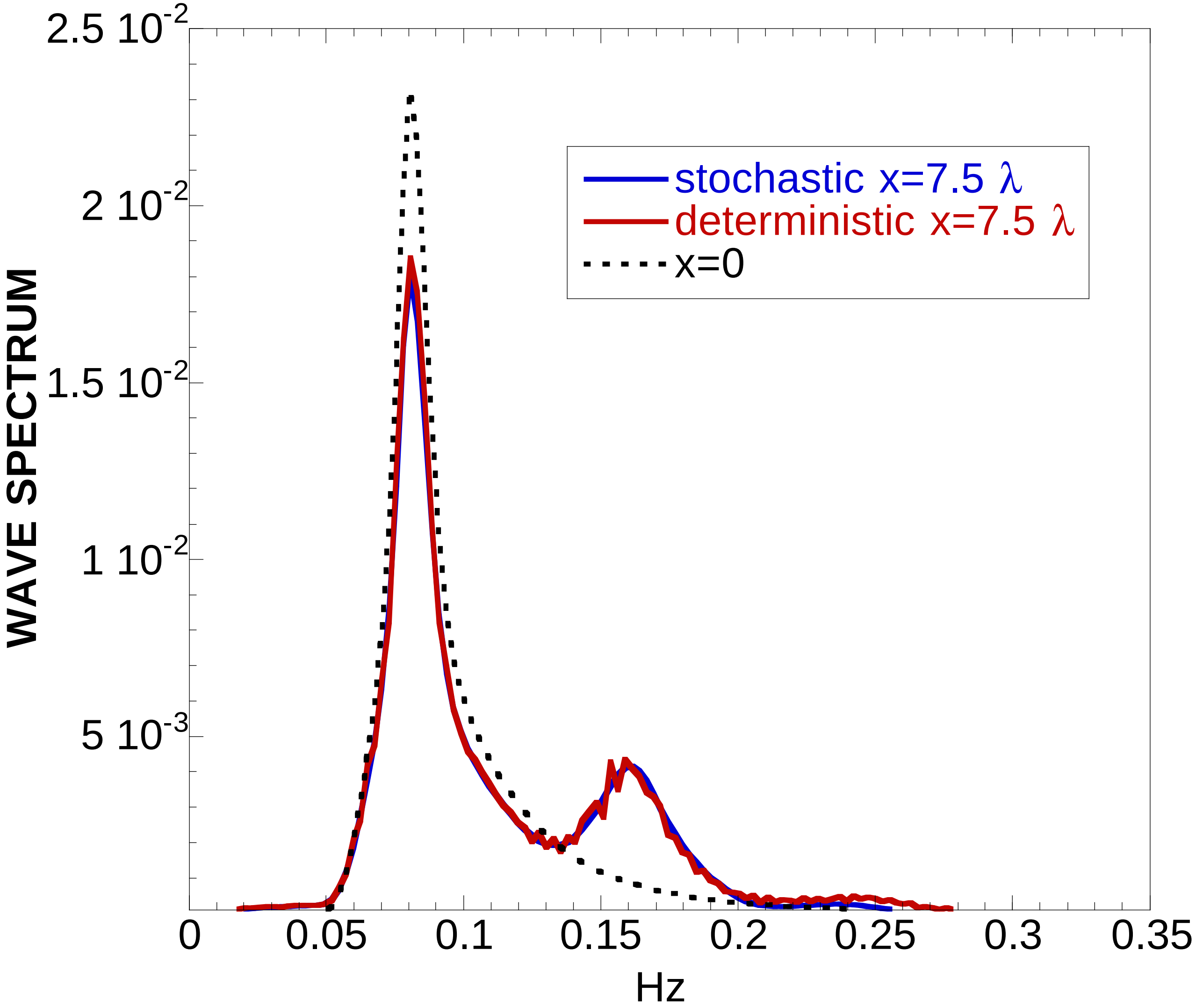
2.5 10-2
2 10-2
1.5 10-2
1 10-2
5 10-3
0 100
0
0.05
0.1
0.15
0.2
0.25
Hz
WAVE SPECTRUM
stochastic x=30.2 λ
deterministic x=30.2 λ
x=0


Skewness
x/λ
Stochastic
Deterministic
0
0.2
0.4
0.6
0.8
1
0
5
10
15
20
25
30

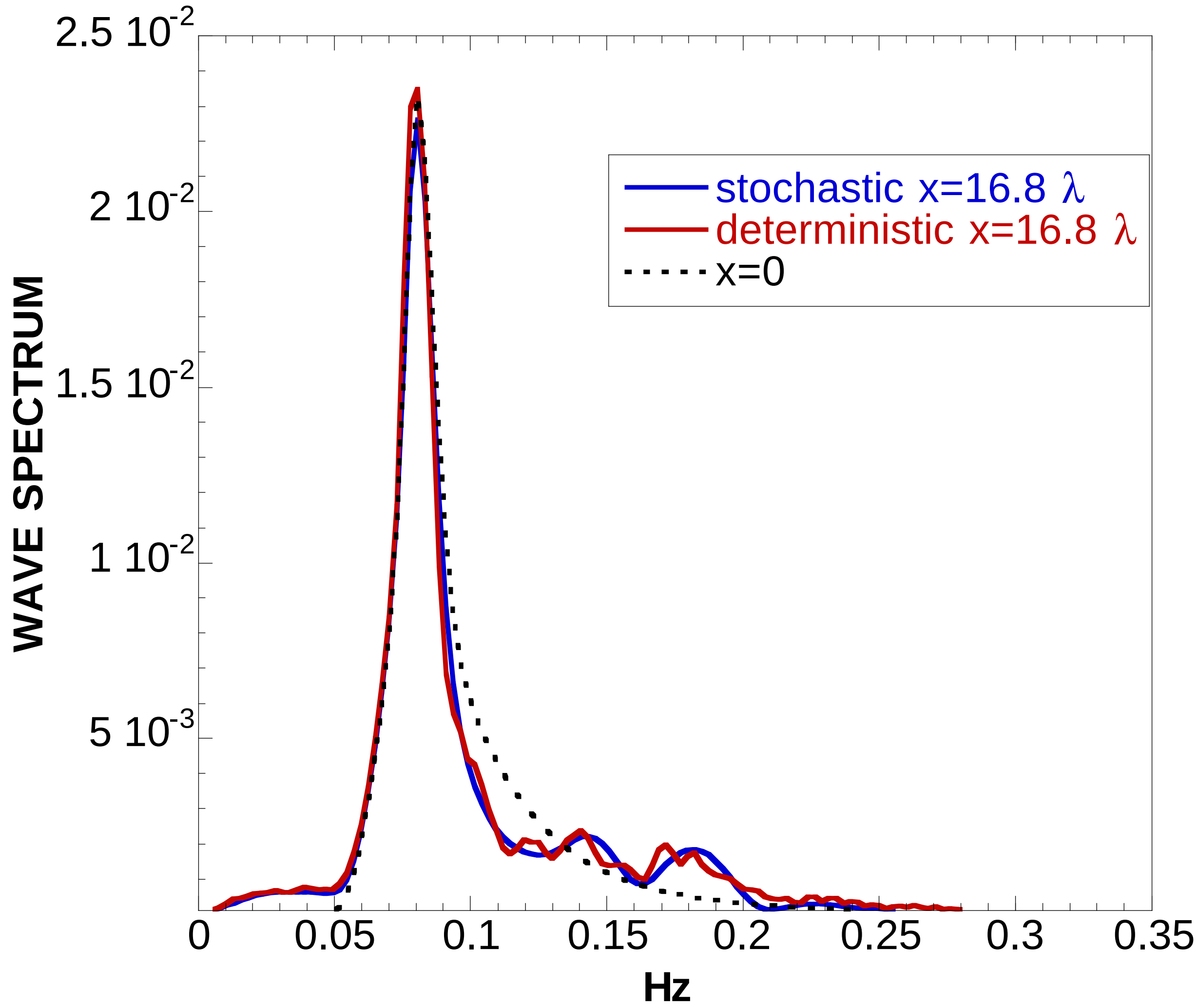
Ur=0.46



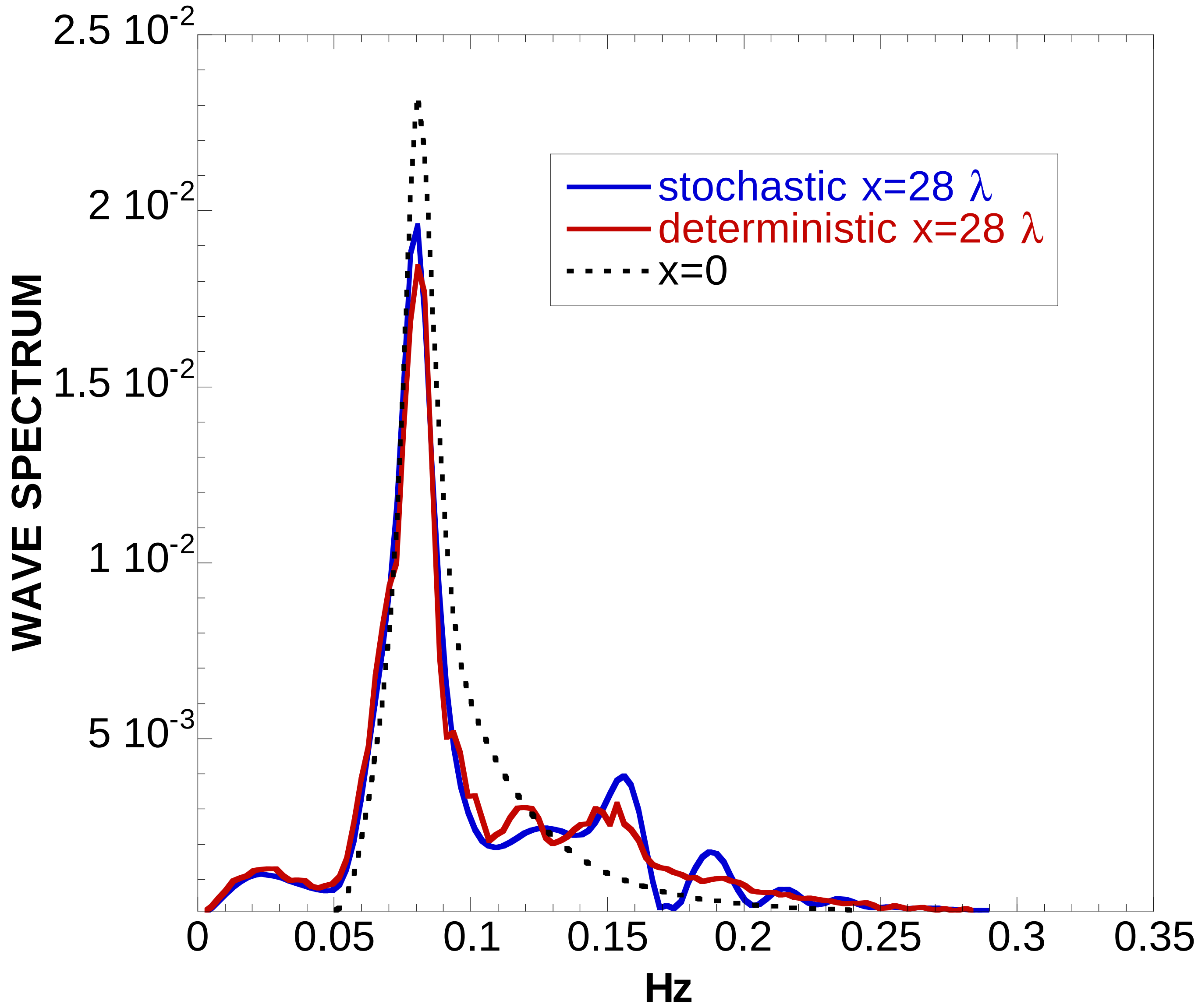
2.5 10-2
2 10-2
1.5 10-2
1 10-2
5 10-3
WAVE SPECTRUM
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
Hz
stochastic x=3.7 λ
deterministic x=3.7 λ
x=0

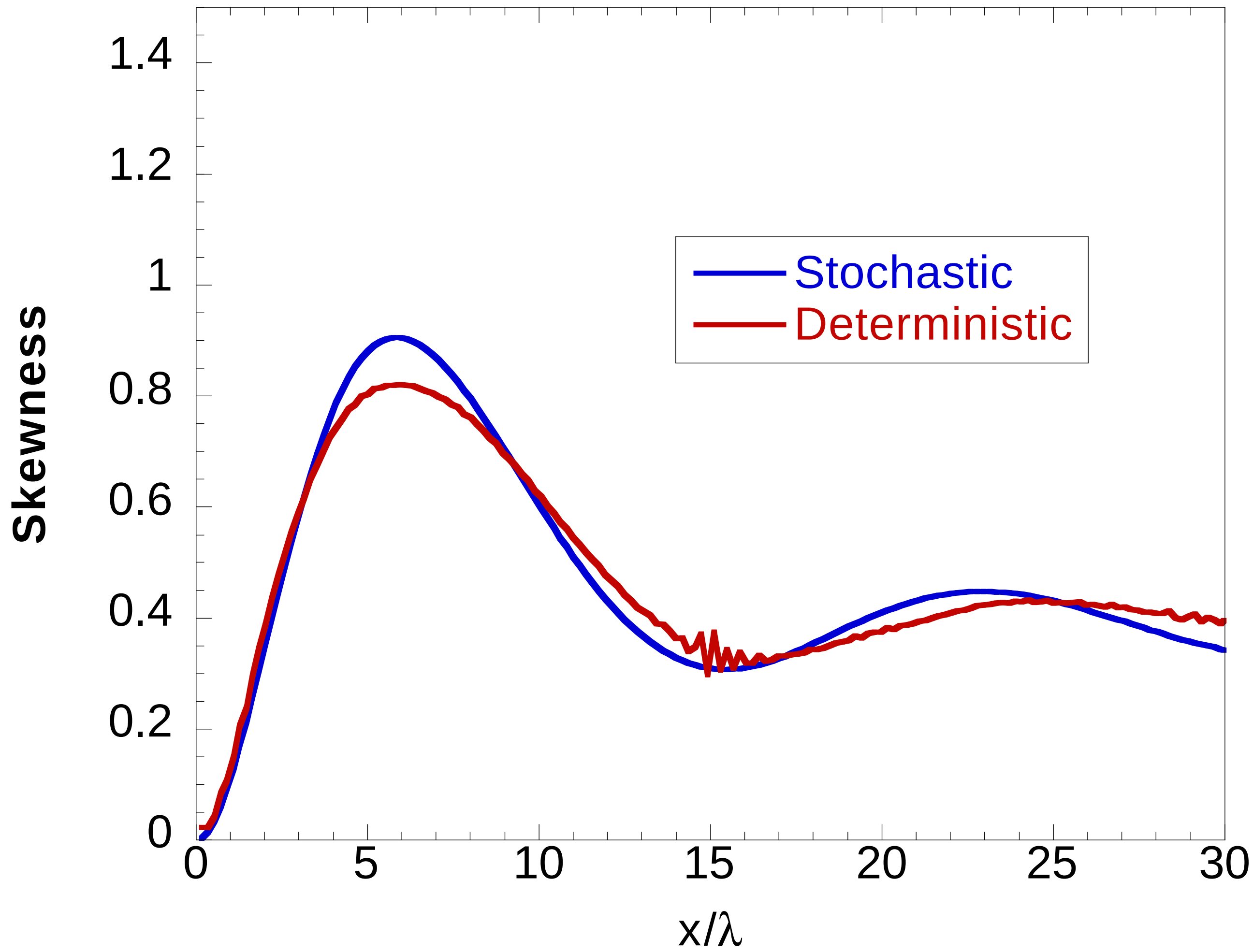


WAVE SPECTRUM
2.5 10-2
2 10-2
1.5 10-2
1 10-2
5 10-3
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
Hz
stochastic x=7.5 λ
deterministic x=7.5 λ
x=0

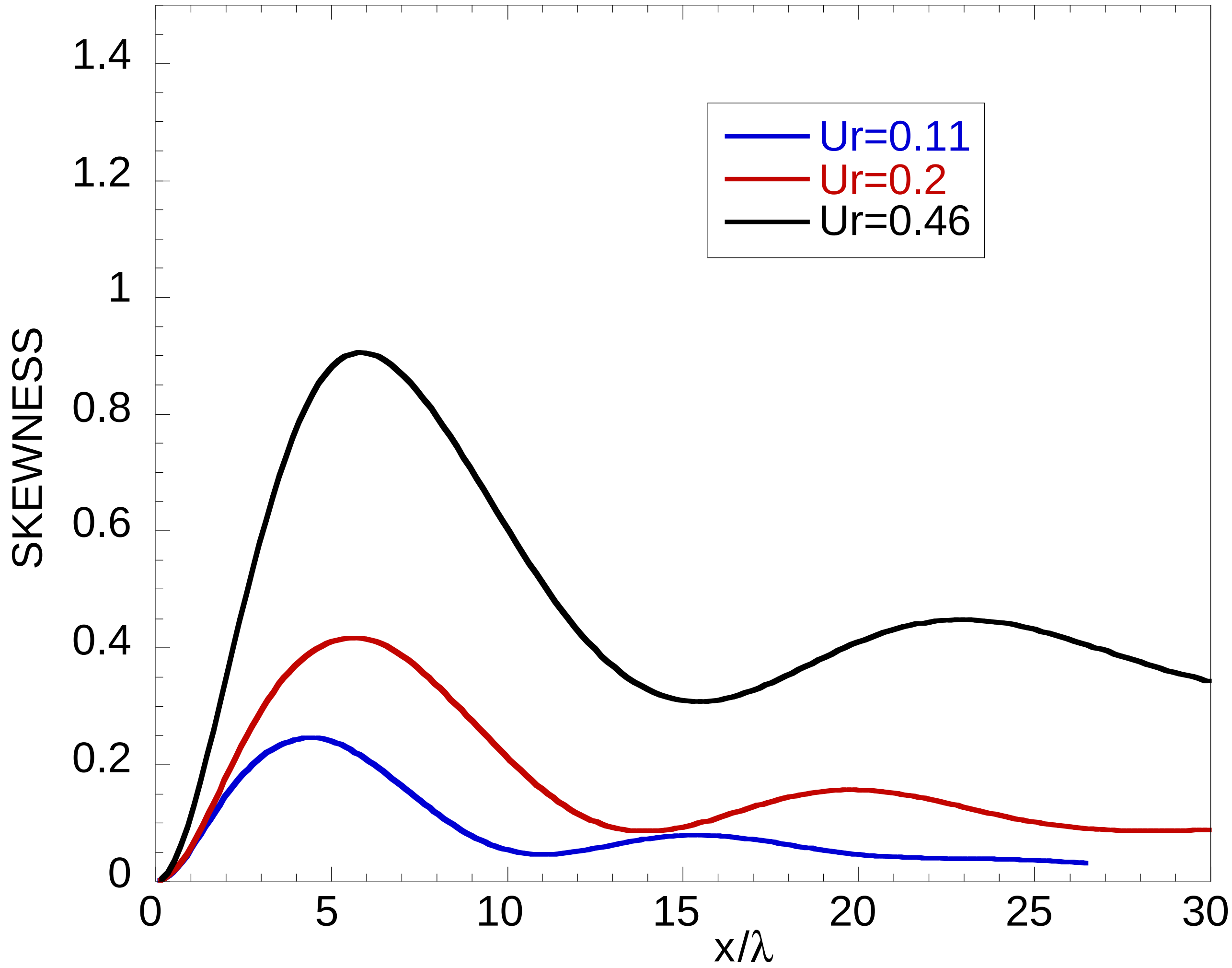


stochastic x=16.8 λ
deterministic x=16.8 λ
x=0
WAVE SPECTRUM
Hz
2.5 10-2
2 10-2
1.5 10-2
1 10-2
5 10-3
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35


2.5 10-2
2 10-2
1.5 10-2
1 10-2
5 10-3
WAVE SPECTRUM
stochastic x=28 λ
deterministic x=28 λ
x=0
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
Hz



Skewness
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
5
10
15
20
25
30
x/λ
Stochastic
Deterministic


Ur=0.11
Ur=0.2
Ur=0.46
SKEWNESS
x/λ
0
0.2
0.4
0.6
0.8
1
1.2
1.4
0
5
10
15
20
25
30

CONCLUSIONS

We have developed a wave action balance equation applicable in shallow water

The equation is based on the most general deterministic equation that includes three wave interactions

In the limit of shallow water, results from the equation have been compared with ensemble numerical simulations of the deterministic equation for different Ursell number in constant depth

For the conditions considered, the main features of the dynamic (including skewness) are well captured by the developed model