SELF-SIMILAR SOLUTIONS OF THE HASSELMANN EQUATION

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Plan:

- Hasselmann equation (HE) and its numerical solution
- Self-similar solutions (*SSS*) for Hasselmann equation (*HE*)
- Numerical confirmation of *SSS* in swell case
- Numerical confirmation of *SSS* in wind-driven case

The Hasselmann equation

$$dn_k / dt = S_{nl} + S_{input} + S_{diss}$$

$$S_{nl} = 2p \int |T_{0123}|^2 (n_0 n_2 n_3 + n_1 n_2 n_3 - n_0 n_1 n_2 - n_0 n_1 n_3)$$

$$\times d(w_0 + w_1 - w_2 - w_3) d(\mathbf{k}_0 + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3$$

Numerical approach: Resio-Tracy code

- Integration around locus defined by:

$$\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}_4$$

$$w_1+w_2=w_3+w_4$$

where $\mathbf{k}_1$ and $\mathbf{k}_3$ are fixed along curves

- Grid resolution 71x36 point in frequency-angle space
- Logarithmic grid in frequency
- Frequency range $0.02Hz < f < 2.0Hz$
- Deep water case
- Wind speed 1m/sec – 20 m/sec
- Wind input cutoff $f_{cutoff} > 1.35Hz$

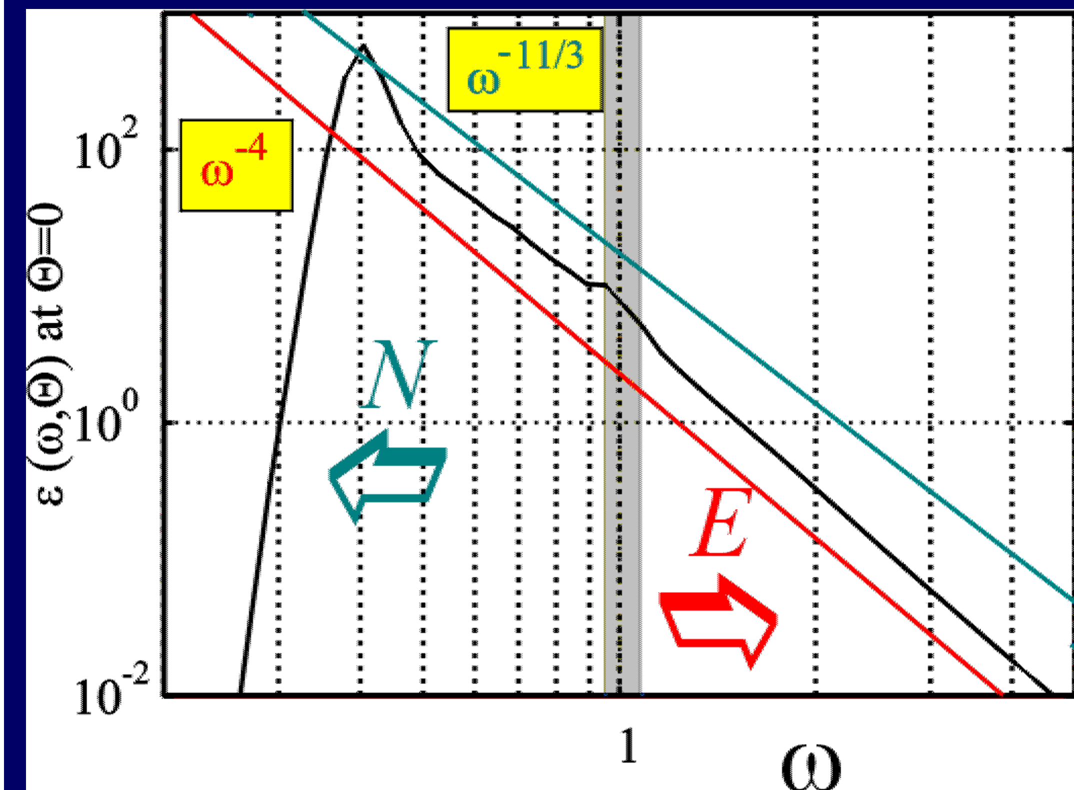
Known issues

- Numerical instability at high frequencies, especially for wind-driven cases
- Small time step, especially for wind-driven cases

The Latest Features:

- Up to 200 times bigger time-steps for wind-driven cases
- Numerical instability-free simulation
- Significantly faster than physical time

Energy and Action Cascades



$$E^{(1)}(w,q) = C_p \frac{g^{4/3}P^{1/3}}{w^4}$$

Direct cascade

Zakharov, Filonenko 1966

$$E^{(2)}(w,q) = C_q \frac{g^{4/3}Q^{1/3}}{w^{11/3}}$$

Inverse cascade

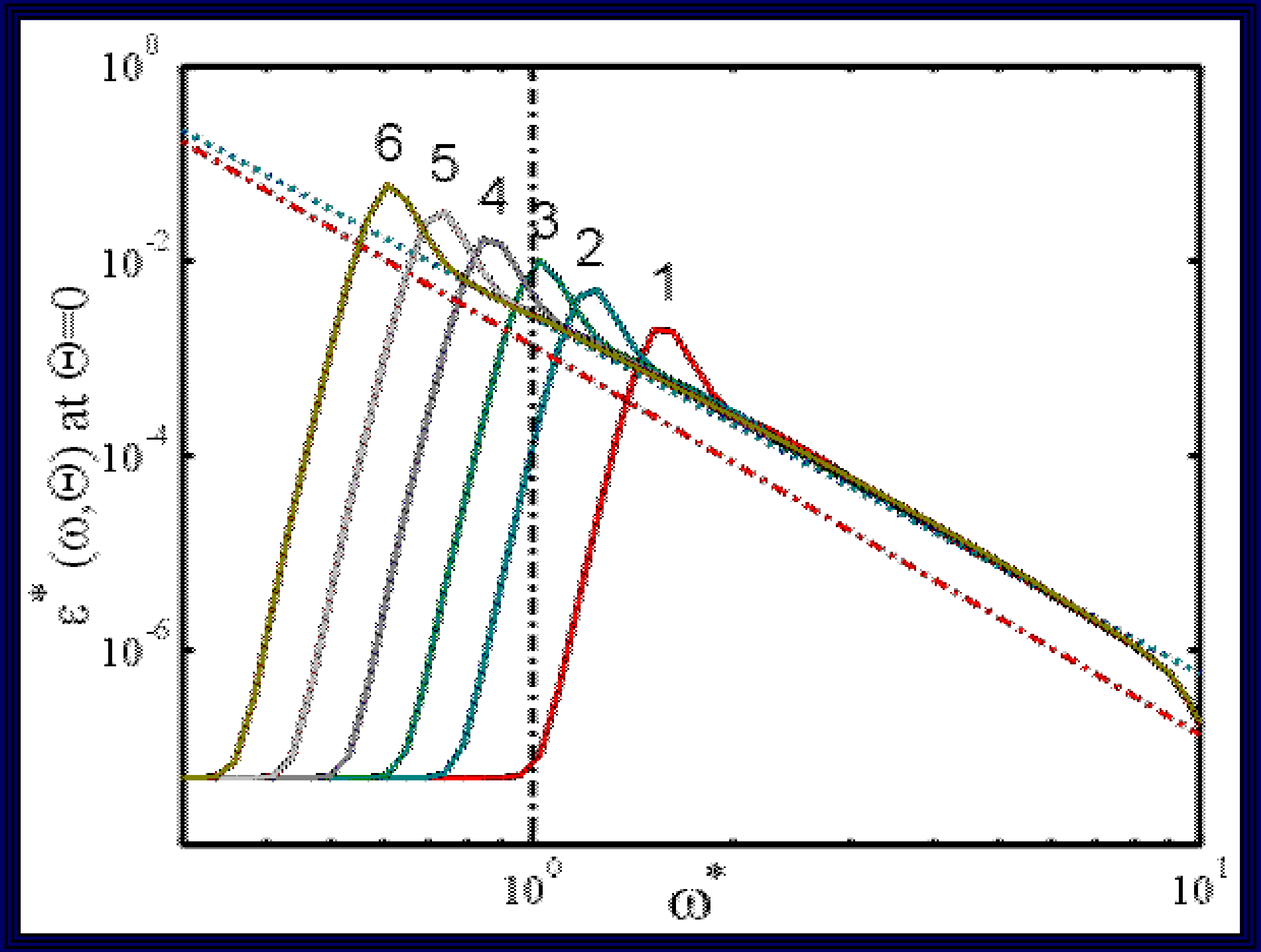
Zakharov, 1966

Zakharov, Zaslavskii 1983

Experimental study by ***Toba, 1972***

Numerical study of Kolmogorov spectra for wind-driven sea
Pushkarev, Resio, Zakharov Physica D 2003

Motivation for studying of self-similarity



Duration-limited growth: times 1, 2, 4, 8, 16, 32 hours (wave input Donelan et al. 1987)

Looking for ***SSS*** in swell situation (no forcing/ dissipation):

$$n = t^{a} U(\mathbf{k} t^{b}); \quad \mathbf{?} = \mathbf{k} t^{b}$$

After substitution into ***HE***:

$$a U_{ß} + ß z \nabla_{?} U_{ß} = t^{2a - \frac{19}{2} b + 1} S_{nl}\left[U_{ß}(?)\right]$$

which means that self-similarity requires

$$2a + 1 - \tfrac{19}{2} b = 0$$

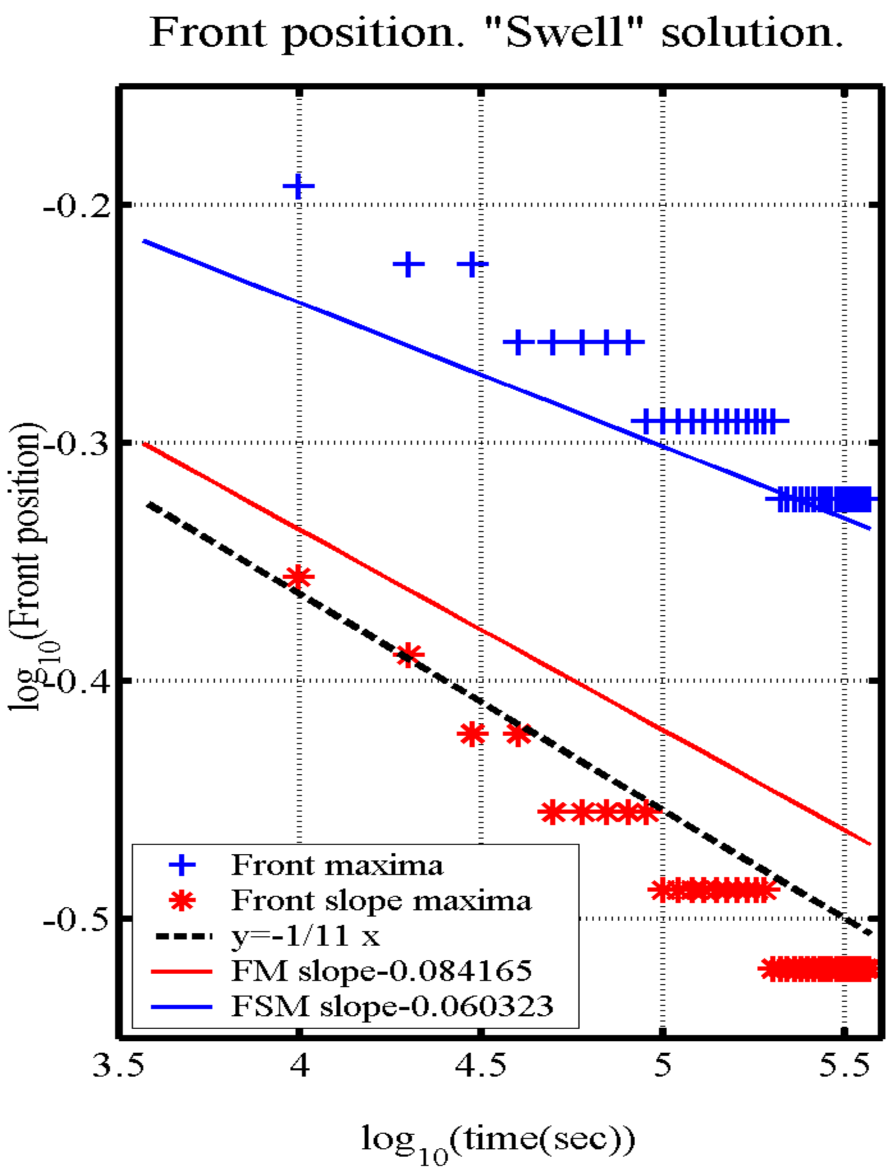
From the condition of stationary wave action for the swell:

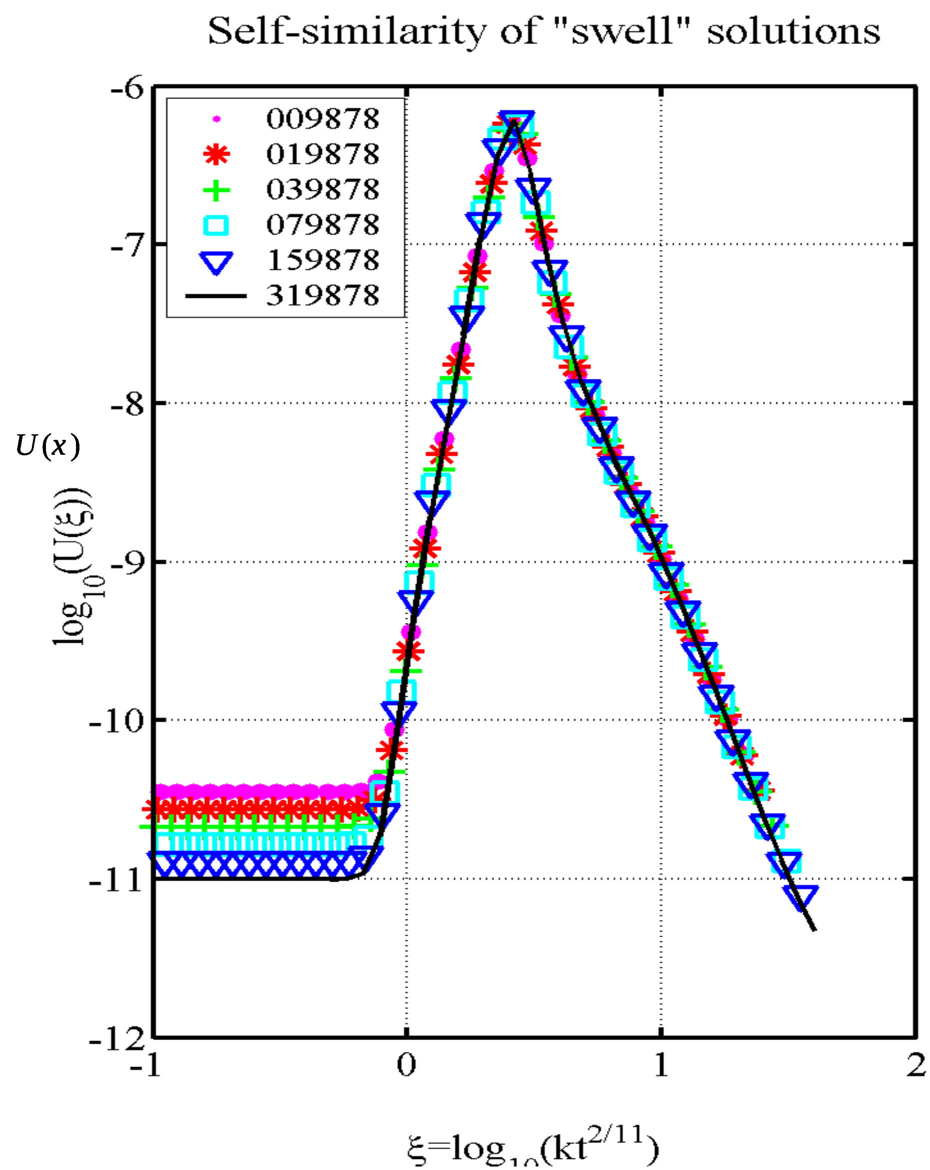
$$N \sim \int t^{a} U(\mathbf{k}t^{b})d\mathbf{k} \sim t^{a-2b} \int U(x)dx$$

$$a = 2b$$

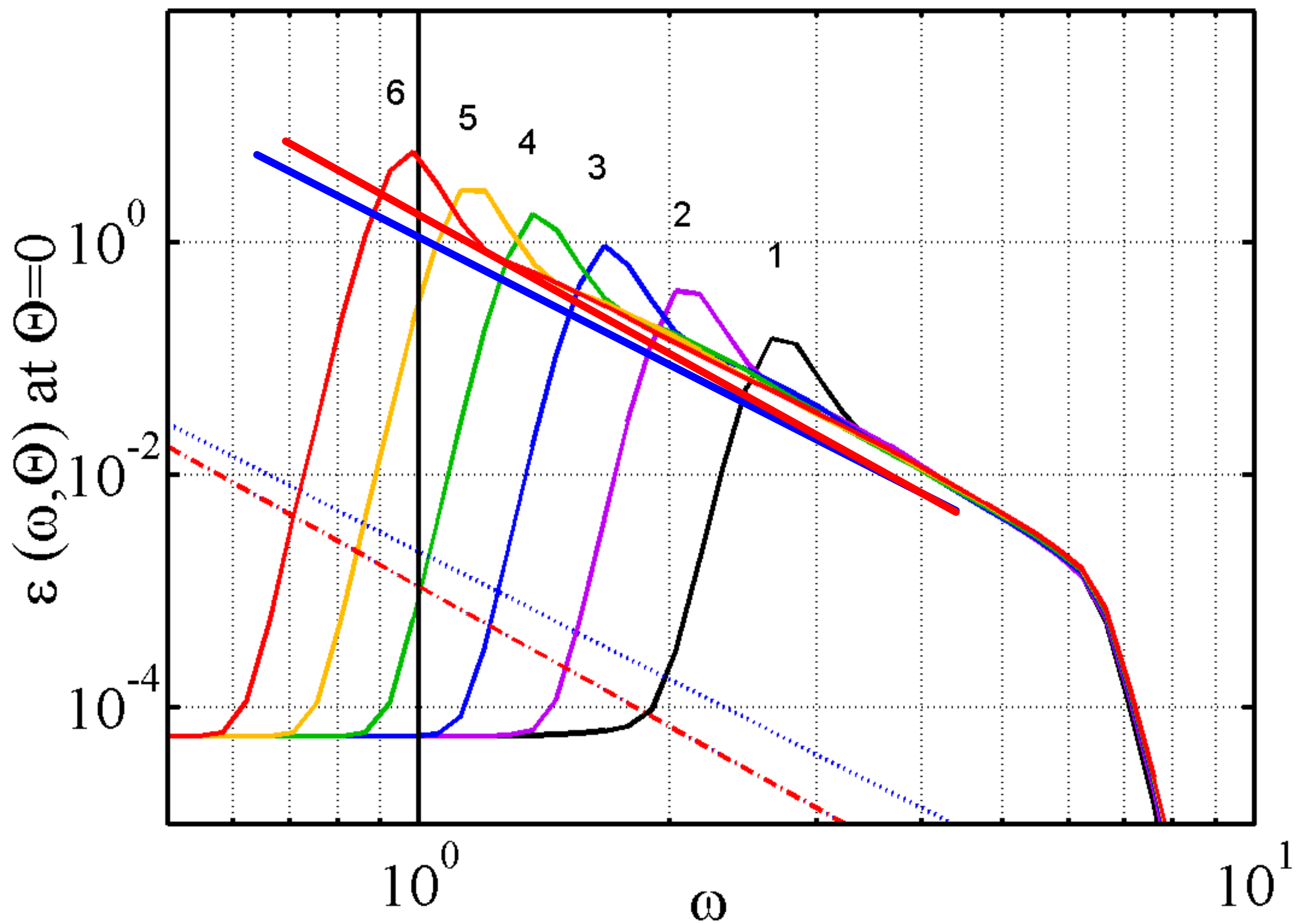
which gives $a = \frac{4}{11}, b = \frac{2}{11}$

Swell solution. Extracting function $U(x)$ from ***SSS*** $n = t^{4/11} U(kt^{2/11})$





Numerical solutions for KE in wind-driven case



U=10m/sec

τ=1, 2, 4, 8, 16, 32 hours

Looking for ***SSS*** in stationary forcing situation

$$n = t^{a} U(\mathbf{k} t^{b}); \quad ? = \mathbf{k} t^{b}$$

After substitution into ***HE***:

$$a U_{ß} + ß z \nabla_{?} U_{ß} = t^{2a - \frac{19}{2} b + 1} S_{nl} \left[U_{ß}(?) \right] + \frac{S_{in} + S_{diss}}{t^{a-1}}$$

which means that if $a \geq 1$, the effect of S_{in} and S_{diss} vanishes as $t \to \infty$ and the same condition

$$2a + 1 - \frac{19}{2} b = 0$$

can provide ***SSS*** solution.

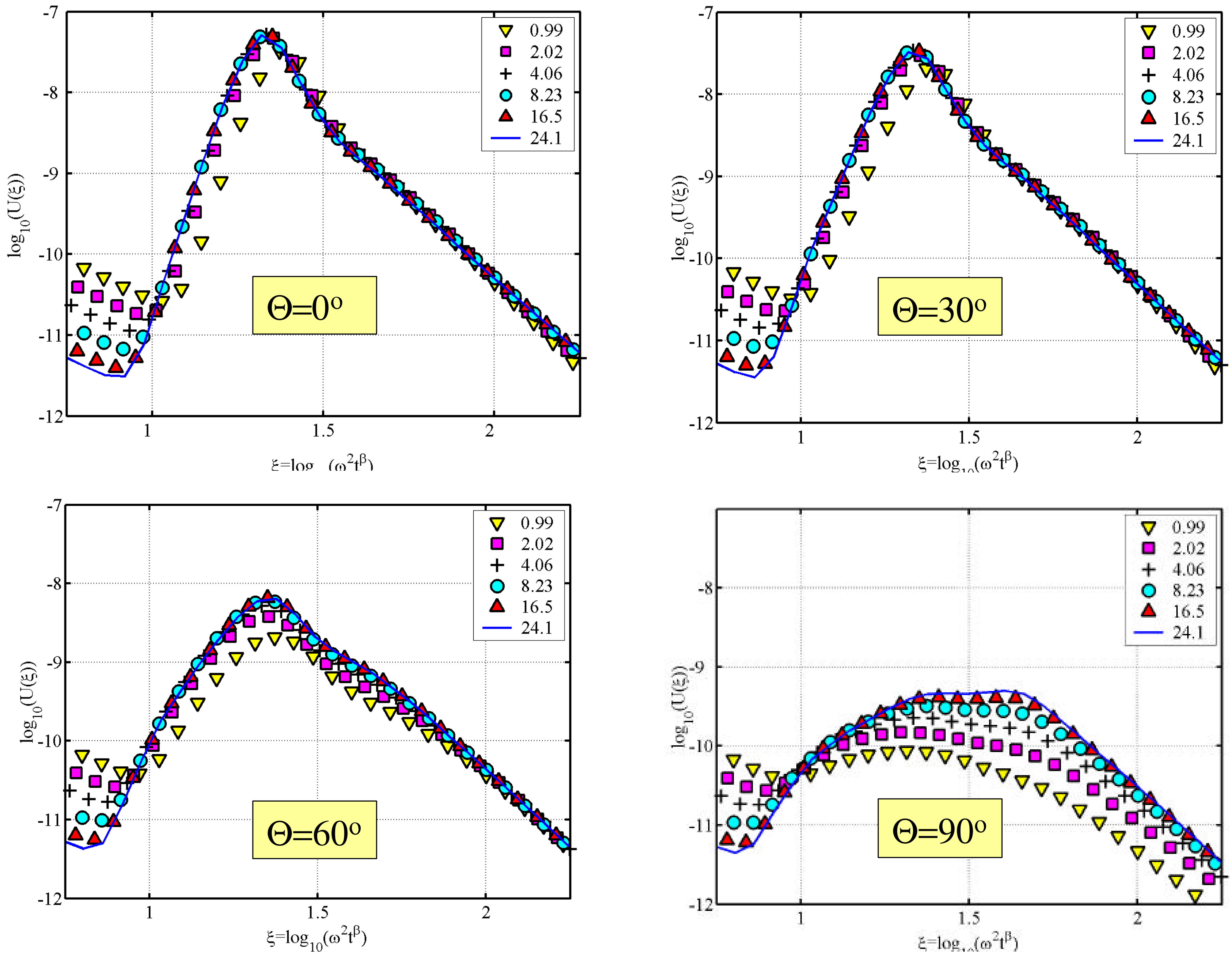
For wind-driven case one can assume that

$$N \sim \int t^{a} U(\mathbf{k}t^{b})d\mathbf{k} \sim t^{a-2b}\int U(x)dx \propto t^{r}$$

and $r = 1$

we get $a = 2b + 1$

So, one can expect $n = t^{23/11}U(kt^{6/11})$


Θ=0°
Θ=30°
Θ=60°
Θ=90°
0.99
2.02
4.06
8.23
16.5
24.1
log10(U(ξ))
ξ=log10(ω²t^β)

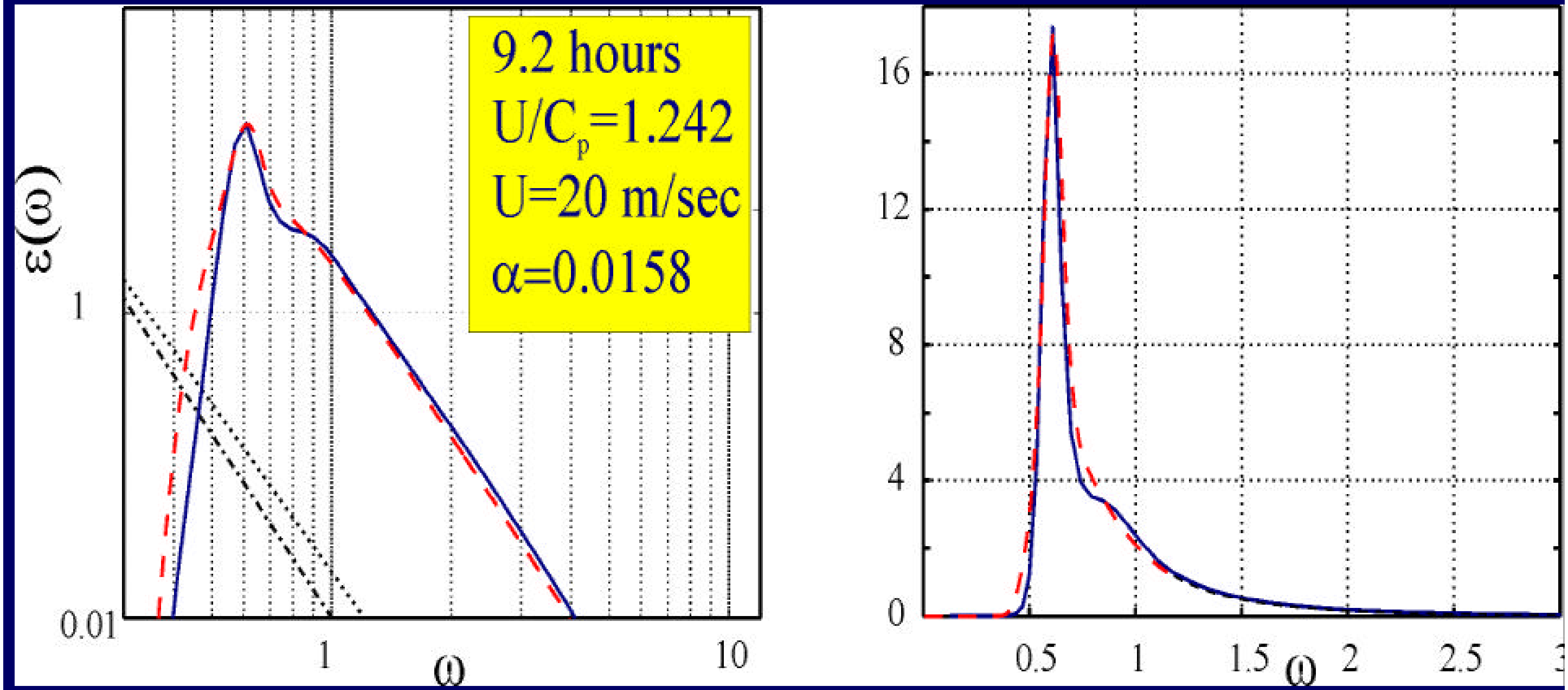
U_{10} =10m/sec	U_{10}=20 m/sec	U_{10}=30 m/sec

Wave input	wave action *r*	Wave energy *p*	Mean freq. *q*	Toba's exponent
Hsiao & Shemdin (1983)	0.91	0.69	0.22	1.57
Stewart (1974) & Plant(1982)	0.99	0.74	0.25	1.48
Donelan (1987)	0.99	0.74	0.25	1.48
Snyder (1981)	0.97	0.73	0.24	1.52
Hsiao & Shemdin (1983)	0.93	0.7	0.23	1.52
Snyder (1981)	1.2	0.9	0.3	1.5
Hsiao & Shemdin (1983)	0.97	0.73	0.24	1.52

Self-similarity of wind wave evolution in experimental studies JONSWAP

$$E(w) = \frac{ag^2}{w_p}\left(\frac{w}{w_p}\right)^{-4} \exp\left(-\left(\frac{w}{w_p}\right)^{-4}\right) g^{\exp\left[-\frac{(w-w_p)^2}{2s^2 w_p^2}\right]}$$

Numerical vs JONSWAP spectrum



U/C_p=1.242

Summary

- There is a strong tendency of numerical solutions of the ***HE*** to self-similar behavior

- These numerical results are consistent with theoretical analysis

- The self-similar solutions can be considered as the extensions of the Kolmogorov-Zakharov cascade solutions

The End